工程數學--微分方程

Differential Equations (DE)

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教學網頁：http://djj.ee.ntu.edu.tw/DE.htm

Chapter 7 The Laplace Transform

作用：把微分變成乘法

Chapter 4 曾經提過 $\frac{d^k}{dt^k}y(t)$ 可寫成 $D^k y(t)$

Laplace transform可以將 $\frac{d^k}{dt^k}y(t)$ 變成

$$s^kY(s)-s^{k-1}y(0)-s^{k-2}y'(0)-\cdots\cdots-sy^{(k-2)}(0)-y^{(k-1)}(0)$$

Section 7-1 Definition of the Laplace Transform

7-1-1 Definitions

- **Laplace Transform** of *f*(*t*)

$$\boxed{L\{f(t)\}=\int_0^{\infty}e^{-st}f(t)d}$$

經常以大寫來代表 transform 的結果

$$F(s)=L\{f(t)\}=\int_0^{\infty}e^{-st}f(t)d$$

Laplace Transform is one of the integral transform

- transform: 把一個 function 變成另外一個 function
- integral transform: 可以表示成積分式的 transform

$$F(s)=\int_a^b \underline{K(s,t)}\, f(t)dt$$

- kernel

對 Laplace transform 而言

$$K(s,t)=e^{-st}, \qquad a=0, \quad b\to\infty$$

註：Chap. 14 將教到的 Fourier transform, 也是一種 integral transform

7-1-2 Linear Property

$$\int_0^{\infty} e^{-st}\left[\alpha f(t)+\beta g(t)\right]dt=\alpha\int_0^{\infty} e^{-st} f(t)dt+\beta\int_0^{\infty} e^{-st} g(t)dt$$

$$L\{\alpha f(t)+\beta g(t)\}=\alpha L\{f(t)\}+\beta L\{g(t)\}$$

事實上，所有的 integral transform 都有 linear property

7-1-3 The Laplace Transforms of Some Basic Functions

$f(t)$	$F(s)$
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
$\exp(at)$	$\frac{1}{s-a}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sinh(kt)$	$\frac{k}{s^2-k^2}$
$\cosh(kt)$	$\frac{s}{s^2-k^2}$

(彼此密切相關)

Example 1 $L\{1\}$

$$L\{1\} = \int_0^\infty e^{-st}dt = -\frac{e^{-st}}{s}\Big|_0^\infty = -\frac{e^{-s\cdot\infty}}{s} - (-\frac{e^{-s\cdot 0}}{s}) = \frac{1}{s}$$

(1) $-\frac{e^{-s\cdot\infty}}{s}$ 比較正式的寫法是 $\lim\limits_{b\to\infty} -\frac{e^{-s\cdot b}}{s}$

(2) 這裡假設 $s > 0$, 所以 $-\frac{e^{-s\cdot\infty}}{s} = 0$

Example 2 $L\{t\}$

$$\int_a^b u(t)v'(t)\,d t\!\!= u(t)v(t)\Big|_a^b - \int_a^b u'(t)v(t)\,d$$

$$L\{t\} = \int_0^\infty t e^{-st}\,dt$$

$$= -\frac{te^{-st}}{s}\Big|_0^\infty + \int_0^\infty \frac{e^{-st}}{s}\,dt$$

$$= -\frac{\infty\cdot e^{-s\cdot\infty}}{s} + \frac{0\cdot e^{-s\cdot 0}}{s} - \frac{e^{-st}}{s^2}\Big|_0^\infty$$

$$= -\frac{e^{-s\cdot\infty}}{s^2} + \frac{e^{-s\cdot 0}}{s^2}$$

$$= \frac{1}{s^2}$$

Example 3 $L\left\{e^{-3t}\right\}=\dfrac{1}{s+3}$

Example 4 $L\{\sin(2t)\}$

除了課本的解法之外，

另一個解法 $\sin(2t)=\dfrac{1}{2i}\left(e^{i2t}-e^{-i2t}\right)$

$$L\{\sin(2t)\}=\frac{1}{2i}L\left\{e^{i2t}\right\}-\frac{1}{2i}L\left\{e^{-i2t}\right\}=\frac{1}{2i}\frac{1}{s-i2}-\frac{1}{2i}\frac{1}{s+i2}$$

$$=\frac{1}{2i}\frac{s+i2-(s-i2)}{(s-i2)(s+i2)}=\frac{1}{2i}\frac{i4}{s^2+4}=\frac{2}{s^2+4}$$

Text page 258 的另外二個例子

$$L\{1+5t\} = L\{1\} + 5L\{t\} = \frac{1}{s} + \frac{5}{s^2}$$

$$L\left\{4e^{-3t} - 10\sin 2t\right\} = 4L\left\{e^{-3t}\right\} - 10L\{\sin 2t\} = \frac{4}{s+3} - \frac{20}{s^2+4}$$

7-1-4 When Does the Laplace Transforms Exist?

Constraint 1 for the existence of the Laplace transform :

For a function $f(t)$, there should exist constants c, $M >$ 0, and $T > 0$ such that

$$|f(t)| \le Me^{ct} \quad \text{for all } t > \mathrm{T}$$

In this condition, $f(t)$ is said to be of exponential order c

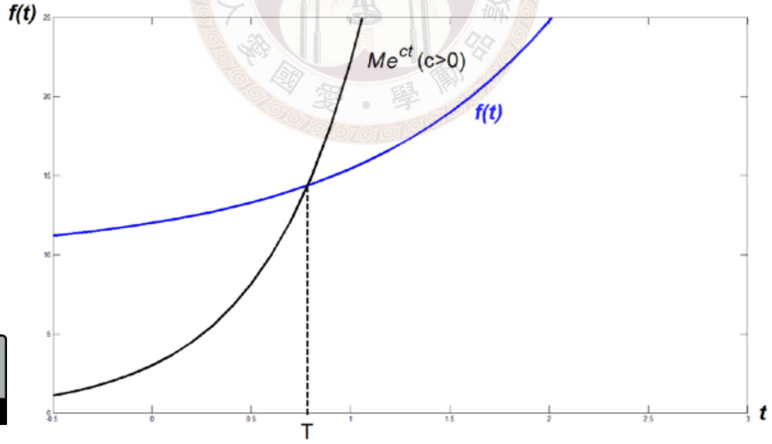
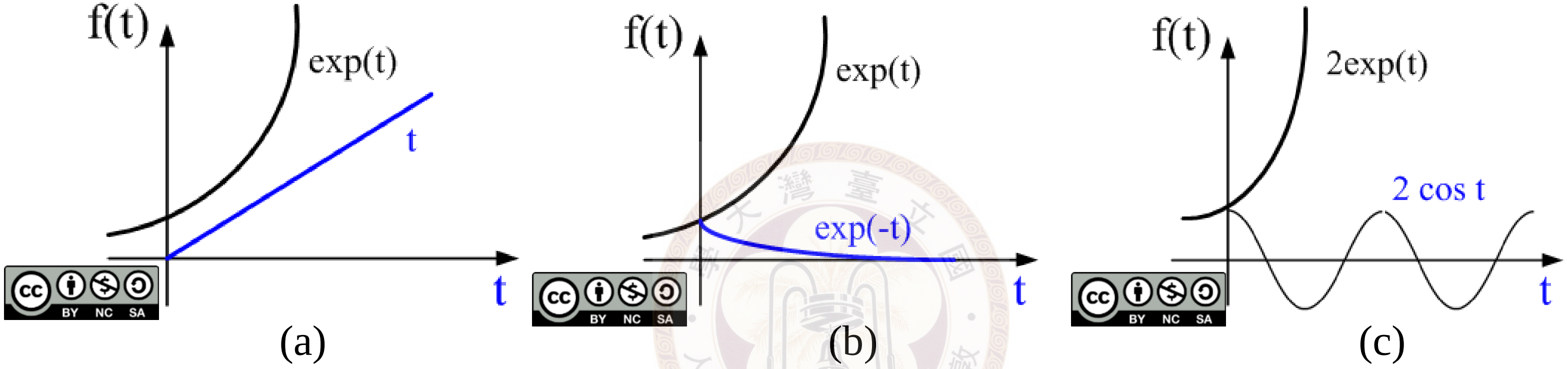


Fig. 7.1.2

例子： $f(t) = t,\ e^{-t}, 2\cos t$ 皆為 exponential order 1

Fig. 7.1.3



(a) (b) (c)

補充：其實，對一個function 而言， exponential order c 不只一個

例子： $f(t) = t^n$ 為 exponential order c, $c > 0$

There exists an M such that $\left|\frac{t^n}{e^{ct}}\right| \le M$ if $c > 0$

例子： $f(t) = \exp(t^2)$ 時，並不存在一個 c 使得

$$\left|f(t)\right| \le Me^{ct} \quad \text{for all } t > T$$

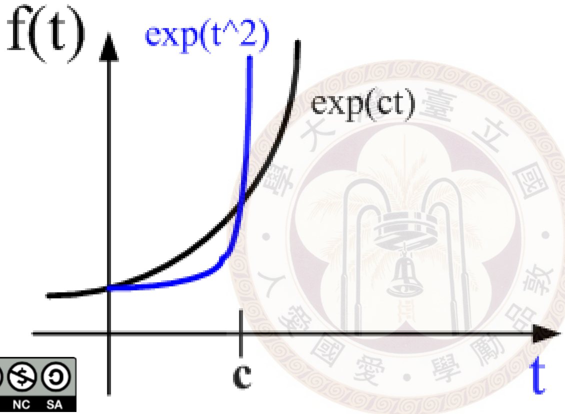


Fig. 7.1.4

只要有一個 c 使得 $\left|f(t)\right| \le Me^{ct} \quad \text{for all } t > T$

我們稱 $f(t)$ 為 of exponential order

否則，我們稱 $f(t)$ 為 not of exponential order

Constraint 2 for the existence of the Laplace transform **:**

$f(t)$ should be piecewise continuous on $[0, \infty)$

在任何 $t \in [a, b]$ 的區間內 $(0 \le a \le b < \infty)$

$f(t)$ 為 discontinuous 的點的個數為有限的

稱作是「piecewise continuous」

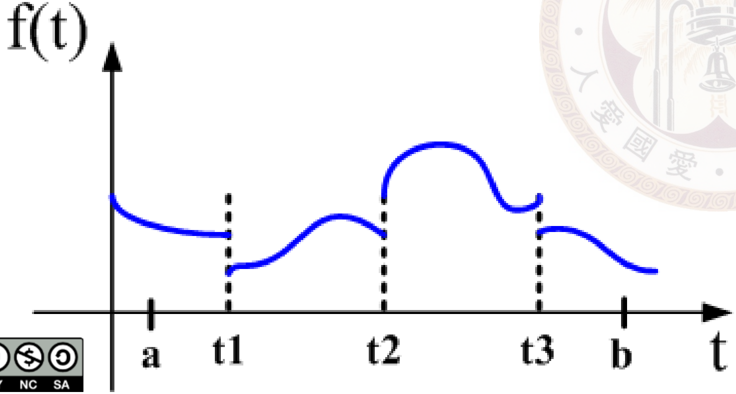


Fig. 7.1.1

注意： $f(t) = 1/t$ 不為 piecewise continuous

Constraints 1 and 2 are “sufficient conditions”

若滿足 → Laplace transform 存在

若不滿足 → Laplace transform 未必不存在

反例： $f(t) = t^{-1/2}$ 不為 piecewise continuous

但是 Laplace transform 存在 $F(s) = s^{-1/2}\sqrt{\pi}$

補充說明：$f(t)$ 不為 piecewise continuous 是因為

$$f(0) \to \infty$$

所以 $f(t)$ 在 $t = 0$ 附近有無限多個不連續點

事實上，只要 $f(t_1) \to \infty$, $|t_1|$ is not infinite，

$f(t)$ 必定不為 piecewise continuous

Theorem 7.1.3

If *f*(*t*) is piecewise continuous on [0, ∞) and of exponential order, then

$$\lim_{s\to\infty} F(s) = 0$$

7-1-5 Section 7-1 需要注意的地方

(1) Laplace transform of some basic functions 要背起來

(2) 記公式時，一些地方要小心 sin, sinh, $1/t^n$

$$\sin kt \longrightarrow \frac{k}{s^2+k^2}$$

沒有平方

有平方

(3) 熟悉(a) 包含 exponential function 的積分

以及 (b) $\int_a^b u(t)v'(t)d\,t = u(t)v(t)\Big|_a^b - \int_a^b u'(t)v(t)d$

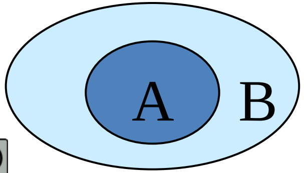
的積分技巧

(4) 要迅速判斷一個式子當 $t \longrightarrow \infty$ 時是否為 0

(5) 小心正負號

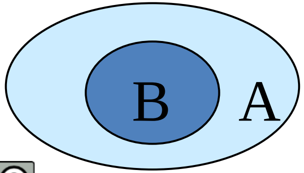
附錄七：充分條件和必要條件的比較

If A is satisfied, then B is also satisfied :



A is the sufficient conditions of B (充分條件)

If B is satisfied, then A is bound to be satisfied :



A is the necessary conditions of B (必要條件)

B is satisfied if and only if A is be satisfied :

A is the necessary and sufficient conditions of B

(充分且必要的條件)

Section 7-2 Inverse Transforms and Transforms of Derivatives

本節有兩大部分：

(1) inverse Laplace transform 的計算 (7-2-1 ~ 7-2-3)

(2) 將微分變成 Laplace transform 當中的乘法 (7-2-4 ~ 7-2-6)

7-2-1 Inverse 方法一： One-to-One Relation

When (1) $f_1(t)$ and $f_2(t)$ are piecewise continuous on $[0, \infty)$, and

(2) $f_1(t)$ and $f_2(t)$ are of exponential order, then

$$\text{if } f_1(t) \neq f_2(t) \longrightarrow \text{then } F_1(s) \neq F_2(s)$$

換句話說，在這種情形下，Laplace transform 是 one-to-one 的運算。

If the Laplace transform of $f_1(t)$ is $F_1(s)$,

then the inverse Laplace transform of $F_1(s)$ must be $f_1(t)$.

Table of Inverse Laplace Transforms

$F(s)$	$L^{-1}\{F(s)\}$
$\frac{1}{s}$	1
$\frac{n!}{s^{n+1}}$	t^{n}
$\frac{1}{s-a}$	$\exp(at)$
$\frac{k}{s^2+k^2}$	$\sin(kt)$
$\frac{s}{s^2+k^2}$	$\cos(kt)$
$\frac{k}{s^2-k^2}$	$\sinh(kt)$
$\frac{s}{s^2-k^2}$	$\cosh(kt)$

Example 1 (text page 263)

(a) $L^{-1}\left\{\frac{1}{s^5}\right\}$

$$L^{-1}\left\{\frac{1}{s^5}\right\}=\frac{1}{4!}L^{-1}\left\{\frac{4!}{s^5}\right\}=\frac{1}{4!}t^4$$

Example 2 (text page 264)

$L^{-1}\left\{\frac{-2s+6}{s^2+4}\right\}$

$$L^{-1}\left\{\frac{-2s+6}{s^2+4}\right\}=L^{-1}\left\{\frac{-2s}{s^2+4}+\frac{6}{s^2+4}\right\}=-2L^{-1}\left\{\frac{s}{s^2+4}\right\}+3L^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$=-2\cos(2t)+3\sin(2t)$$

7-2-2 Inverse 方法 (二） Decomposition of Fractions

Example 3 (text page 264) $L^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\}$

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+4}$$

$$L^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\} = Ae^t + Be^{2t} + Ce^{-4t}$$

問題：A, B, C 該如何算出？

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = \frac{A(s-2)(s+4)+B(s-1)(s+4)+C(s-1)(s-2)}{(s-1)(s-2)(s+4)}$$

$$s^2+6s+9=(A+B+C)s^2+(2A+3B-3C)s-8A-4B+2C$$ 太麻煩

7-2-3 計算分數分解係數的快速法

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}=\frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s+4}$$

兩邊各乘上 $(s-1)$

$$\frac{s^2+6s+9}{(s-2)(s+4)}=A+(s-1)\frac{B}{s-2}+(s-1)\frac{C}{s+4}$$

把 $s=1$ 代入 $-\frac{16}{5}=A$

這二個步驟可以合併

左式乘上 $(s-2)$後，把 $s=2$ 代入 $B=\left.\frac{s^2+6s+9}{(s-1)\cancel{(s-2)}(s+4)}\right|_{s=2}=\frac{25}{6}$

左式乘上 $(s+4)$後，把 $s=-4$ 代入 $C=\left.\frac{s^2+6s+9}{(s-1)(s-2)\cancel{(s+4)}}\right|_{s=-4}=\frac{1}{30}$

通則：要將一個 fraction 分解　　(Cover up method)

$$\frac{K(s)}{(s-a_1)(s-a_2)\cdots\cdots(s-a_N)}=Q(s)+\frac{A_1}{s-a_1}+\frac{A_2}{s-a_2}+\cdots\cdots+\frac{A_N}{s-a_N}$$

$a_1, a_2, \ldots., a_N$ 互異

(1) 用多項式的除法算出 $Q(s)$

商　　　　餘式

$$\frac{K(s)}{(s-a_1)(s-a_2)\cdots\cdots(s-a_N)}=\underset{\uparrow\,商}{Q(s)}+\frac{\overset{餘式\,\searrow}{K_1(s)}}{(s-a_1)(s-a_2)\cdots\cdots(s-a_N)}$$

使得 order of $K_1(s) < N$

(2) 算出 A_n

$$A_n=\left.\frac{K_1(s)}{(s-a_1)(s-a_2)\cdots\cdots(s-a_{n-1})\cancel{(s-a_n)}(s-a_{n+1})\cdots\cdots(s-a_N)}\right|_{s=a_n}$$

例子：$\boxed{\dfrac{s^4-8s^3+31s^2-36s+20}{(s-1)(s-2)(s-3)^2}=Q(s)+\dfrac{A_1}{s-1}+\dfrac{A_2}{s-2}+\dfrac{A_3+A_4(s-3)}{(s-3)^2}}$

$$\frac{s^4-8s^3+31s^2-36s+20}{(s-1)(s-2)(s-3)^2}=1+\frac{s^3+2s^2+3s+2}{(s-1)(s-2)(s-3)^2}$$

$$A_1=\left.\frac{s^3+2s^2+3s+2}{\cancel{(s-1)}(s-2)(s-3)^2}\right|_{s=1}=-\frac{8}{4}=-2$$

$$A_2=\left.\frac{s^3+2s^2+3s+2}{(s-1)\cancel{(s-2)}(s-3)^2}\right|_{s=2}=24$$

$$\frac{s^3+2s^2+3s+2}{(s-1)(s-2)}=(s-3)^2\frac{A_1}{s-1}+(s-3)^2\frac{A_2}{s-2}+A_3+A_4(s-3)$$

$$A_3=\left.\frac{s^3+2s^2+3s+2}{(s-1)(s-2)}\right|_{s=3}=\frac{56}{2}=28$$

$$\frac{s^3+2s^2+3s+2}{(s-1)(s-2)}=(s-3)^2\frac{A_1}{s-1}+(s-3)^2\frac{A_2}{s-2}+A_3+A_4(s-3)$$

$$A_4=\frac{d}{ds}\frac{s^3+2s^2+3s+2}{(s-1)(s-2)}\bigg|_{s=3}=\frac{(3s^2+4s+3)(s-1)(s-2)-(s^3+2s^2+3s+2)(2s-3)}{(s-1)^2(s-2)^2}\bigg|_{s=3}$$

$$=-21$$

$$\boxed{\frac{s^4-8s^3+31s^2-36s+20}{(s-1)(s-2)(s-3)^2}=1-\frac{2}{s-1}+\frac{24}{s-2}+\frac{28-21(s-3)}{(s-3)^2}}$$

- 小技巧：其實，如果只剩下一個未知數，我們可以將 s 用某個數代入， 快速的將未知數解出

例如，前面的例子，將 $s = 0$ 代入原式

$$\frac{2}{18} = -A_1 - \frac{A_2}{2} + \frac{A_3 - 3A_4}{9} \qquad A_4 = (-1 - 9A_1 - \frac{9A_2}{2} + A_3)/3 = -21$$

例子：$\boxed{\dfrac{s^2+2s+3}{(s-1)(s^2+2s+2)} = \dfrac{A_1}{s-1} + \dfrac{A_2 s + A_3}{s^2+2s+2}}$

$$A_1 = \left.\frac{s^2+2s+3}{\cancel{(s-1)}(s^2+2s+2)}\right|_{s=1} = \frac{6}{5}$$

$$\frac{s^2+2s+3}{(s-1)(s^2+2s+2)} - \frac{6/5}{s-1} = \frac{1}{5}\frac{-s^2-2s+3}{(s-1)(s^2+2s+2)} = \frac{1}{5}\frac{-s-3}{s^2+2s+2}$$

$$\boxed{\frac{s^2+2s+3}{(s-1)(s^2+2s+2)} = \frac{6/5}{s-1} - \frac{1}{5}\frac{s+3}{s^2+2s+2}}$$

7-2-4 Transforms of Derivatives

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)d \;= \underline{te^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)d}$$

$$= 0 - f(0) + sL\{f(t)\} = sL\{f(t)\} - f(0)$$

$$\int_a^b u(t)v'(t)d\;t\!= u(t)v(t)\Big|_a^b - \int_a^b u'(t)v(t)d$$

$$L\{f''(t)\} = sL\{f'(t)\} - f'(0) = s\left[sL\{f(t)\} - f(0)\right] - f'(0)$$

$$= s^2L\{f(t)\} - sf(0) - f'(0)$$

$$L\{f'''(t)\} = sL\{f''(t)\} - f''(0) = s^3L\{f(t)\} - s^2 f(0) - sf'(0) - f''(0)$$

Theorem 7.2.2 Derivative Property of the Laplace Transform

$$L\left\{f^{(n)}(t)\right\}=s^{n}F(s)-s^{n-1}f(0)-s^{n-2}f'(0)-\cdots\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$$

7-2-5 Solving the Constant Coefficient Linear DE by Laplace Transforms

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y = g(x)$$

$$\Big\downarrow \text{ Laplace transform}$$

$$\boxed{\begin{aligned}
&a_n\left[s^n Y(s) - s^{n-1} y(0) - s^{n-2} y'(0) - \cdots\cdots - s y^{(n-2)}(0) - y^{(n-1)}(0)\right] \\
&+ a_{n-1}\left[s^{n-1} Y(s) - s^{n-2} y(0) - s^{n-1} y'(0) - \cdots\cdots - s y^{(n-3)}(0) - y^{(n-2)}(0)\right] \\
&+ \cdots\cdots \\
&+ a_1\left[sY(s) - y(0)\right] + a_0 Y(s) = G(s)
\end{aligned}}$$

$$\boxed{\begin{aligned}&a_n\left[s^nY(s)-s^{n-1}y(0)-s^{n-2}y'(0)-\cdots\cdots-sy^{(n-2)}(0)-y^{(n-1)}(0)\right]\\&+a_{n-1}\left[s^{n-1}Y(s)-s^{n-2}y(0)-s^{n-1}y'(0)-\cdots\cdots-sy^{(n-3)}(0)-y^{(n-2)}(0)\right]\\&+\cdots\cdots\\&+a_1\left[sY(s)-y(0)\right]+a_0Y(s)=G(s)\end{aligned}}$$

$$\boxed{P(s)Y(s)-Q(s)=G(s)}$$

$$P(s)=a_ns^n+a_{n-1}s^{n-1}+\cdots\cdots a_1s+a_0$$

(auxiliary)

$$\begin{aligned}Q(s)=\;&a_n\left[s^{n-1}y(0)+s^{n-2}y'(0)+\cdots\cdots+sy^{(n-2)}(0)+y^{(n-1)}(0)\right]\\&+a_{n-1}\left[\qquad s^{n-2}y(0)+\cdots\cdots+sy^{(n-3)}(0)+y^{(n-2)}(0)\right]\\&+\cdots\cdots\\&+a_2\left[\qquad\qquad sy(0)\quad+\quad y'(0)\right]\\&+a_1\left[\qquad\qquad\qquad y(0)\right]\end{aligned}$$

$$Y(s)=\frac{Q(s)}{P(s)}+\frac{G(s)}{P(s)} \qquad Y(s)=W(s)Q(s)+W(s)G(s)$$

$$W(s)=\frac{1}{P(s)}$$

$G(s)$: Laplace transform of the input

$Q(s)$: caused by initial conditions

$Y(s)$: Laplace transform of the response

$W(s)$: transform function

$L^{-1}[W(s)Q(s)]$: zero-input response or state response

$L^{-1}[W(s)G(s)]$: zero-state response or input response

Example 4 (text page 266)

$$y'(t)+3y(t)=13\sin 2t \qquad y(0)=6$$

$$sY(s)-y(0)+3Y(s)=13\frac{2}{s^2+4}$$

$$(s+3)Y(s)=6+\frac{26}{s^2+4}$$

$$Y(s)=\frac{6}{s+3}+\frac{26}{(s+3)(s^2+4)}$$

$$\left.\frac{26}{\cancel{(s+3)}(s^2+4)}\right|_{s=-3}=\frac{26}{13}=2$$

$$Y(s)=\frac{8}{s+3}+\frac{-2s+6}{s^2+4}$$

$$\frac{26}{(s+3)(s^2+4)}-\frac{2}{s+3}$$

$$=\frac{-2s^2+18}{(s+3)(s^2+4)}=\frac{-2s+6}{s^2+4}$$

$$Y(s)=\frac{8}{s+3}+-2\frac{s}{s^2+4}+3\frac{2}{s^2+4}$$

$$y(t)=8e^{-3t}-2\cos 2t+3\sin 2t$$

Example 5 (text page 267)

$$y''(t)-3y'(t)+2y(t)=e^{-4t} \qquad y(0)=1 \qquad y'(0)=5$$

$$\Big\downarrow \text{快速法}$$

$$(s^2-3s+2)Y(s)=s+2+\frac{1}{s+4}$$

$$\begin{aligned}
Y(s)&=\frac{s+2}{s^2-3s+2}+\frac{1}{(s^2-3s+2)(s+4)}\\
&=\frac{s+2}{(s-1)(s-2)}+\frac{1}{(s-1)(s-2)(s+4)}\\
&=-\frac{3}{s-1}+\frac{4}{s-2}-\frac{1}{5}\frac{1}{s-1}+\frac{1}{6}\frac{1}{s-2}+\frac{1}{30}\frac{1}{s+4}\\
&=-\frac{16}{5}\frac{1}{s-1}+\frac{25}{6}\frac{1}{s-2}+\frac{1}{30}\frac{1}{s+4}
\end{aligned}$$

$$y(t)=-\frac{16}{5}e^{t}+\frac{25}{6}e^{2t}+\frac{1}{30}e^{-4t}$$

7-2-6 快速法

(A) 求 $P(s)$

$$\frac{d^k}{dt^k} \to s^k$$

$$y''(t) - 3y'(t) + 2y(t) \to P(s) = s^2 - 3s + 2$$

很像 Sec. 4-3 的…

(B) 求 $Q(s)$

$$\begin{aligned} Q(s) = {} & a_n\left[s^{n-1}y(0) + s^{n-2}y'(0) + \cdots\cdots + sy^{(n-2)}(0) + y^{(n-1)}(0)\right] \\ & + a_{n-1}\left[s^{n-2}y(0) + \cdots\cdots + sy^{(n-3)}(0) + y^{(n-2)}(0)\right] \\ & + \cdots\cdots \\ & + a_2\left[sy(0) + y'(0)\right] \\ & + a_1\left[y(0)\right] \end{aligned}$$

$$\begin{array}{rccccc}
 & s^{n-1} & s^{n-2} & \cdots & s & 1 \\
a_n \times & y(0) & y'(0) & \cdots & y^{(n-2)}(0) & y^{(n-1)}(0) \\
a_{n-1} \times & & y(0) & \cdots & y^{(n-3)}(0) & y^{(n-2)}(0) \\
\vdots & & & \ddots & \vdots & \vdots \\
a_2 \times & & & & y(0) & y'(0) \\
a_1 \times & & & & & y(0) \\
\hline
\end{array}$$

相加

例如，page 444 的例子

$$\begin{array}{rcc}
 & s & 1 \\
1\times & 1 & 5 \\
-3\times & & 1 \\
\hline
\end{array}
=
\begin{array}{cc}
s & 1 \\
1 & 5 \\
 & -3 \\
\hline
s & 2
\end{array}
\Longrightarrow Q(s)$$

7-2-7 Section 7.2 需要注意的地方

(1) 熟悉分數分解

(2) 可以簡化運算的方法，能學則學

鼓勵各位同學多發揮創意，多多研究能簡化計算的快速法

數學上.......並沒有標準解法的存在

(3) Derivative 公式 initial conditions 的順序別弄反

$$L\left\{f^{(n)}(t)\right\}=s^{n}F(s)-s^{n-1}f(0)-s^{n-2}f'(0)-\cdots\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$$

Section 7-3 Operational Properties I

介紹兩個可以簡化 Laplace transform 計算的重要性質

First Translation Theorem (translation for *s*)

$$\boxed{L\left\{e^{at}f\left(t\right)\right\}=F\left(s-a\right)}$$

Second Translation Theorem (translation for *t*)

$$\boxed{L\left\{f\left(t-a\right)u\left(t-a\right)\right\}=e^{-as}F\left(s\right)}$$

$u(t)$: step function

(注意兩者之間的異同)

7-3-1 First Translation Theorem (Translation for *s*)

$$\boxed{L\left\{e^{at}f(t)\right\}=F(s-a)}$$

Proof:

$$L\left\{e^{at}f(t)\right\}=\int_0^\infty e^{-st}e^{at}f(t)dt=\int_0^\infty e^{-(s-a)t}f(t)dt=F(s-a)$$

7-3-1-1 Inverse of "Translation for *s*"

When $f(t)$ is piecewise continuous and of exponential order

$$\boxed{L^{-1}\left\{F\left(s-a\right)\right\}=e^{at}f\left(t\right)}$$ (一對一)

註： Sections 7-3 和 7-4 其他的定理亦如此

7-3-1-2 Examples

Example 1 (text page 271)

(a) $$L\left\{e^{5t}t^{3}\right\} = L\left\{t^{3}\right\}\Big|_{s\to s-5} = \frac{3!}{s^{4}}\Bigg|_{s\to s-5} = \frac{6}{(s-5)^{4}}$$

(b) $$L\left\{e^{-2t}\cos 4t\right\} = L\left\{\cos 4t\right\}\Big|_{s\to s-(-2)} = \frac{s}{s^{2}+16}\Bigg|_{s\to s+2} = \frac{s+2}{(s+2)^{2}+16}$$

Example 2 (text page 272)

(a) $$L^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\} = L^{-1}\left\{\frac{2(s-3)+11}{(s-3)^2}\right\} = 2L^{-1}\left\{\frac{1}{s-3}\right\} + 11L^{-1}\left\{\frac{1}{(s-3)^2}\right\}$$

$$= 2e^{3t} + 11t\,e^{3t}$$

(b) $$L^{-1}\left\{\frac{s/2+5/3}{s^2+4s+6}\right\} = L^{-1}\left\{\frac{s/2+5/3}{(s+2)^2+2}\right\} = L^{-1}\left\{\frac{(s+2)/2+2/3}{(s+2)^2+2}\right\}$$

$$= \frac{1}{2}L^{-1}\left\{\frac{s+2}{(s+2)^2+2}\right\} + \frac{\sqrt{2}}{3}L^{-1}\left\{\frac{\sqrt{2}}{(s+2)^2+2}\right\}$$

$$= \frac{1}{2}e^{-2t}\cos\sqrt{2}t + \frac{\sqrt{2}}{3}e^{-2t}\sin\sqrt{2}t$$

Example 3 (text page 273)

$$y'' - 6y' + 9y = t^2 e^{3t} \qquad y(0) = 2, \quad y'(0) = 17$$

$$\begin{array}{rrr} 1\times & 2 & 17 \\ -6\times & & 2 \end{array} = \begin{array}{rr} 2 & 17 \\ & -12 \\ \hline 2s & 5 \end{array}$$

$$L\left\{t^2 e^{3t}\right\} = L\left\{t^2\right\}\Big|_{s \to s-3} = \frac{2}{s^3}\Big|_{s \to s-3} = \frac{2}{(s-3)^3}$$

$$(s^2 - 6s + 9)Y(s) = 2s + 5 + \frac{2}{(s-3)^3}$$

$$(s^2-6s+9)Y(s)=2s+5+\frac{2}{(s-3)^3}$$

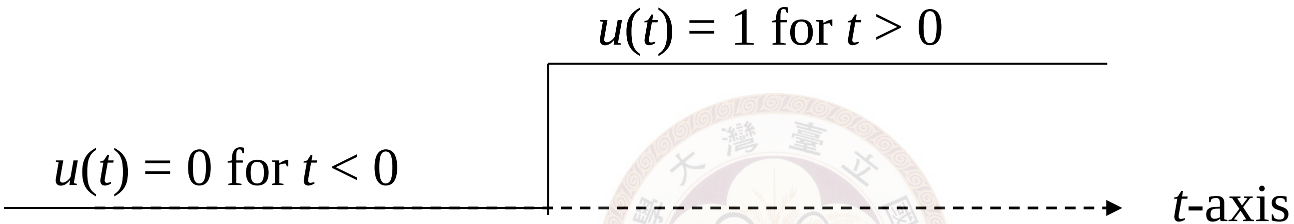
$$Y(s)=\frac{2s+5}{(s-3)^2}+\frac{2}{(s-3)^5}=\frac{2(s-3)+11}{(s-3)^2}+\frac{2}{(s-3)^5}$$

$$=\frac{2}{s-3}+\frac{11}{(s-3)^2}+\frac{1}{12}\frac{4!}{(s-3)^5}$$

$$y(t)=2e^{3t}+11te^{3t}+\frac{1}{12}t^4e^{3t}$$

7-3-2 Step Function

$u(t)$: unit step function



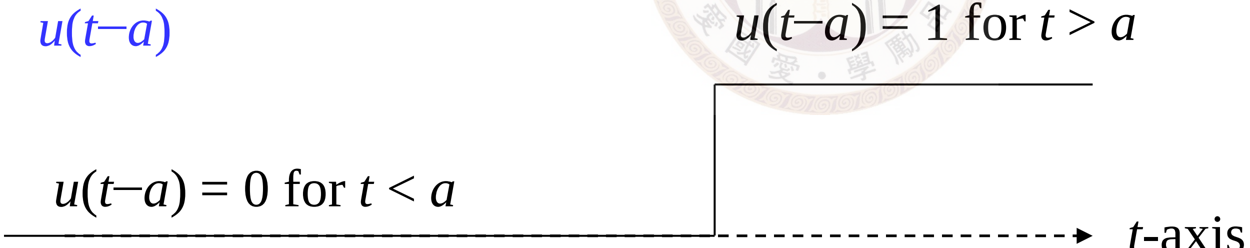
$u(t) = 1$ for $t > 0$

$u(t) = 0$ for $t < 0$

t-axis

$t = 0$

$u(t-a)$



$u(t-a) = 1$ for $t > a$

$u(t-a) = 0$ for $t < a$

t-axis

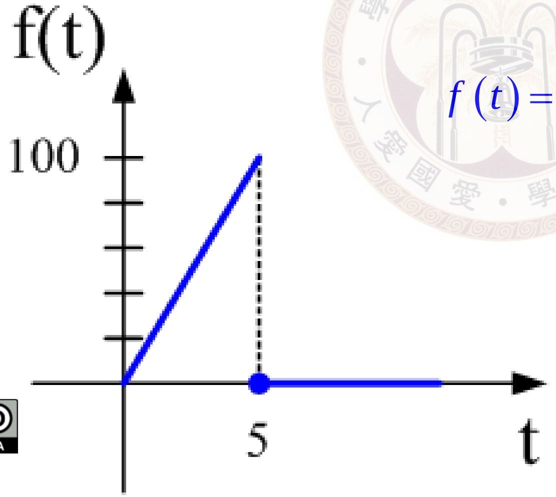
$t = a$

The unit step function acts as a switch (開關).

- Any piecewise continuous function can be expressed as the unit step function for $t \geq 0$

Example 5 (text page 275)

$$f(t) = \begin{cases} 20t & \text{for } 0 \leq t < 5 \\ 0 & \text{for } t > 5 \end{cases}$$



$$f(t) = 20t \cdot u(t) - 20t \cdot u(t-5)$$

In general,

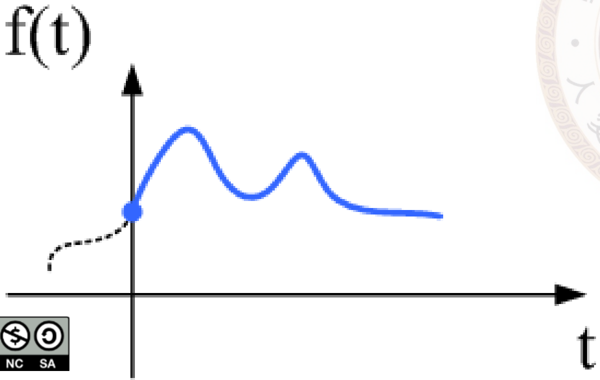
$$f(t)=\begin{cases} h_1(t) & \text{for } 0 \le t < a \\ h_2(t) & \text{for } t > a \end{cases}$$

$$f(t)=h_1(t)\cdot u(t)+(h_2(t)-h_1(t))\cdot u(t-a)$$

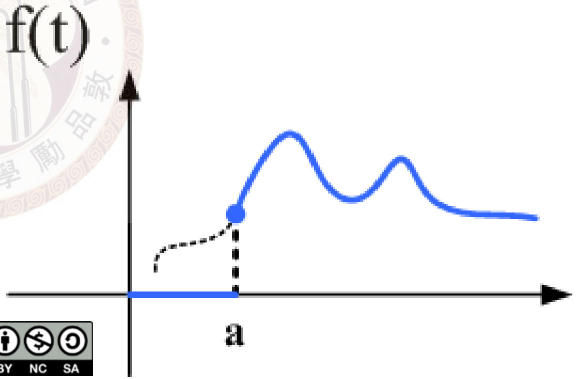
7-3-3 Second Translation Theorem (Translation for *t*)

$$\boxed{L\{f(t-a)u(t-a)\}=e^{-as}F(s)}\qquad a>0$$

或 $$\boxed{L\{g(t)u(t-a)\}=e^{-as}L\{g(t+a)\}}\qquad a>0$$



(a) f(t), t≥0



(b) f(t-a)u(t-a)

Proof:

$$L\{f(t-a)u(t-a)\} = \int_0^\infty e^{-st} f(t-a)u(t-a)\,dt = \int_a^\infty e^{-st} f(t-a)\,dt$$

$$= \int_0^\infty e^{-s(t_1+a)} f(t_1)\,dt_1 \longleftarrow \text{令 } t_1 = t-a$$

$$= e^{-as}\int_0^\infty e^{-st_1} f(t_1)\,dt_1 = e^{-as}F(s)$$

Example 6 (text page 276) $\boxed{L^{-1}\left\{\frac{1}{s-4}e^{-2s}\right\}}$

$$L^{-1}\left\{\frac{1}{s-4}\right\}=e^{4t}, \qquad L^{-1}\left\{\frac{1}{s-4}e^{-2s}\right\}=e^{4(t-2)}u(t-2)$$

Example 7 (text page 276) $\boxed{L\{\cos t\cdot u(t-\pi)\}}$

$$L\{\cos(t+\pi)\}=-L\{\cos(t)\}=-\frac{s}{s^2+1}$$

$$L\{\cos t\cdot u(t-\pi)\}=-\frac{s}{s^2+1}e^{-\pi s}$$

Example 8 (text page 277)

$$y' + y = f(t) \qquad y(0) = 5 \qquad f(t) = \begin{cases} 0 & \text{for } 0 \le t < \pi \\ 3\cos t & \text{for } t \ge \pi \end{cases}$$

$$f(t) = 3\cos t \cdot u(t-\pi)$$

$$L\{\cos(t+\pi)\} = -L\{\cos(t)\} = -\frac{s}{s^2+1}$$

$$L\{3\cos t \cdot u(t-\pi)\} = -\frac{3s}{s^2+1}e^{-\pi s}$$

$$(s+1)Y(s) = 5 - 3\frac{3s}{s^2+1}e^{-\pi s}$$

$$Y(s) = \frac{5}{s+1} - \frac{3}{2}\left[-\frac{1}{s+1} + \frac{1}{s^2+1} + \frac{s}{s^2+1}\right]e^{-\pi s}$$

$$Y(s)=\frac{5}{s+1}-\frac{3}{2}\left[\underline{-\frac{1}{s+1}+\frac{1}{s^2+1}+\frac{s}{s^2+1}}\right]e^{-\pi s}$$

$$\downarrow$$

$$-e^{-t}+\sin(t)+\cos(t)$$

$$y(t)=5e^{-t}+\frac{3}{2}\left[e^{-(t-\pi)}-\sin(t-\pi)-\cos(t-\pi)\right]u(t-\pi)$$

$$=5e^{-t}+\frac{3}{2}\left[e^{-(t-\pi)}+\cos(t)+\sin(t)\right]u(t-\pi)$$

7-3-4 本節需要注意的地方

(1) 套用 “translation for *t*” 的公式時，

先將 input 變成 $g(t+a)$ 再作 Laplace transform

(例如 Example 7)

(2) 變成「標準型態」後再作 inverse

(3) Second translation theorem (translation for *t*) 當 $a > 0$ 時才適用

(4) 套用公式時，注意「順序」

Section 7-4 Operational Properties II

7-4-1 Derivatives of Transforms

$$\boxed{L\left\{t^n f(t)\right\} = (-1)^n \frac{d^n}{ds^n} F(s)}$$

比較：

$$L\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots\cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

微分 $\xrightarrow{\text{Laplace}}$ 乘 s^n

乘 t^n $\xrightarrow{\text{Laplace}}$ 微分

Proof of the Theorem of Derivatives of Transforms:

$$F(s)=\int_0^\infty e^{-st}f(t)dt$$

$$\frac{d}{ds}F(s)=\frac{d}{ds}\int_0^\infty e^{-st}f(t)dt=\int_0^\infty \frac{d}{ds}[e^{-st}]f(t)dt=-\int_0^\infty e^{-st}tf(t)dt=-L\left\{tf(t)\right\}$$

$$\frac{d^n}{ds^n}F(s)=\frac{d^n}{ds^n}\int_0^\infty e^{-st}f(t)dt=\int_0^\infty \frac{d^n}{ds^n}[e^{-st}]f(t)dt=\int_0^\infty e^{-st}(-t)^n f(t)dt$$

$$=L\left\{(-t)^n f(t)\right\}=(-1)^n L\left\{t^n f(t)\right\}$$

Example 1 (text page 283) $\boxed{L\{t\sin kt\}}$

$$L\{\sin kt\}=\frac{k}{s^2+k^2} \qquad L\{t\sin kt\}=-\frac{d}{ds}\frac{k}{s^2+k^2}=\frac{2ks}{(s^2+k^2)^2}$$

練習：為何 $L\{t\cos kt\}=\dfrac{s^2-k^2}{(s^2+k^2)^2}$

7-4-2 Convolution (旋積)

Definition of convolution:

$$\boxed{f(t)*g(t)=\int_{-\infty}^{\infty}f(\tau)g(t-\tau)d\tau}$$ (標準定義)

* 代表 convolution

$$\boxed{f(t)*g(t)=\int_{0}^{t}f(\tau)g(t-\tau)d\tau}$$ (課本關於 convolution 的定義)

When $f(t) = 0$ for $t < 0$ and $g(t) = 0$ for $t < 0$,

上方的式子可以簡化為下方的式子

Convolution 的物理意義 (重要)

當 $y(t)=\int_0^t f(\tau)g(t-\tau)d\tau$

Input $f(\tau)$ 對 output $y(t)$ 的影響為 $g(t-\tau)$

$g(t-\tau)$ 只和 t 與 τ 之間的差有關

Input $f(\tau)$ 對 output $y(t)$ 的影響，決定於 t 與 τ 之間的差

例如： $f(\tau)$ 是在 τ 這個時間點上太陽照射到某個地方的熱量
$g(t\text{-}\tau)$ 可想像成是經過了 $t\text{-}\tau$ 的時間之後，還未幅射回外太熱量比例
$y(t)$ 可想像成是溫度

7-4-3 Convolution Theorem

$$\boxed{L\{f(t)*g(t)\}=L\{f(t)\}L\{g(t)\}=F(s)G(s)}$$

Convolution → Multiplication

Proof: $F(s)G(s)=\left(\int_0^\infty e^{-s\tau}f(\tau)d\tau\right)\left(\int_0^\infty e^{-s\beta}g(\beta)d\beta\right)$

$$=\int_0^\infty\int_0^\infty e^{-s(\tau+\beta)}f(\tau)g(\beta)d\beta d\tau$$

note (A)
見後頁說明

$\downarrow$ 令 $t=\tau+\beta$

note (B)
見後頁說明

$$=\int_0^\infty\int_\tau^\infty e^{-st}f(\tau)g(t-\tau)dtd\tau$$

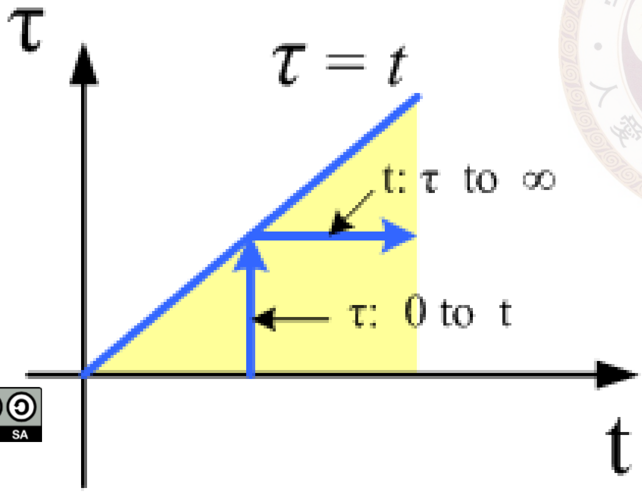
$$=\int_0^\infty e^{-st}[\int_0^t f(\tau)g(t-\tau)d\tau]dt=L[f*g]$$

note (A)

定理： $\iint \cdots\cdots\cdots dxdy = \iint \cdots\cdots\cdots C^{-1} dwdv$ $\qquad C = \left|\det\begin{bmatrix} \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix}\right|$

note (B)

積分範圍的改變：



$$\left|\det\begin{bmatrix} \dfrac{\partial t}{\partial \beta} & \dfrac{\partial t}{\partial \tau} \\ \dfrac{\partial \tau}{\partial \beta} & \dfrac{\partial \tau}{\partial \tau} \end{bmatrix}\right| = \left|\det\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right| = 1$$

Example 3 (text page 284)

$$L\left\{\int_0^t e^{\tau}\sin(t-\tau)d\tau\right\} = L\left\{e^t * \sin t\right\} = \frac{1}{s-1}\frac{1}{s^2+1}$$

Example 4 (text page 285)

$$L^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\} = \frac{1}{k^2}\left(\sin kt * \sin kt\right) = \frac{1}{k^2}\int_0^t \sin k\tau \sin k(t-\tau)d\tau$$

7-4-4 Integration

$$L\left\{\int_0^t f(\tau)\,d\tau\right\} = L\{f(t)*1\} = \frac{F(s)}{s}$$

(想成 “負一次微分”)

Example:

$$L^{-1}\left\{\frac{1}{s(s^2+1)}\right\}=\int_0^t \sin\tau\, d\tau=-\cos t+1$$

Example:

$$L_1\frac{di(t)}{dt}+Ri(t)+\frac{Q(t)}{C}=E(t)$$

$$L_1\frac{di(t)}{dt}+Ri(t)+\frac{Q(0)}{C}+\frac{1}{C}\int_0^t i(\tau)\,d\tau=E(t)$$

$$L_1 sI(s)-L_1 i(0)+RI(s)+\frac{Q(0)}{C}\frac{1}{s}+\frac{1}{C}\frac{I(s)}{s}=L\{E(t)\}$$

7-4-5 Transform of a Periodic Function

Theorem 7.4.3 When $f(t+T) = f(t)$

then $$L\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

Proof: 令 $f_1(t) = f(t)$ when $0 \le t < T$

$f_1(t) = 0$ otherwise

$$f(t) = f_1(t) + f_1(t-T) + f_1(t-2T) + f_1(t-3T) + \ldots\ldots\ldots\ldots\ldots$$

$$= f_1(t) + f_1(t-T)u(t-T) + f_1(t-2T)\,u(t-2T) + f_1(t-3T)\,u(t-3T)$$

$$+ \ldots\ldots\ldots\ldots\ldots$$

$$L\{f(t)\} = L\{f_1(t)\} + L\{f_1(t-T)u(t-T)\} + L\{f_1(t-2T)\,u(t-2T)\}$$

$$+L\{f_1(t-3T)\,u(t-3T)\} + \ldots\ldots\ldots\ldots\ldots\ldots$$

$$L\{f_1(t)\} = \int_0^T e^{-st} f_1(t)\,dt$$

$$L\{f_1(t-T)u(t-T)\} = e^{-sT} L\{f_1(t)\}$$

$$L\{f_1(t-2T)u(t-2T)\} = e^{-2sT} L\{f_1(t)\}$$

$$L\{f_1(t-3T)u(t-3T)\} = e^{-3sT} L\{f_1(t)\}$$

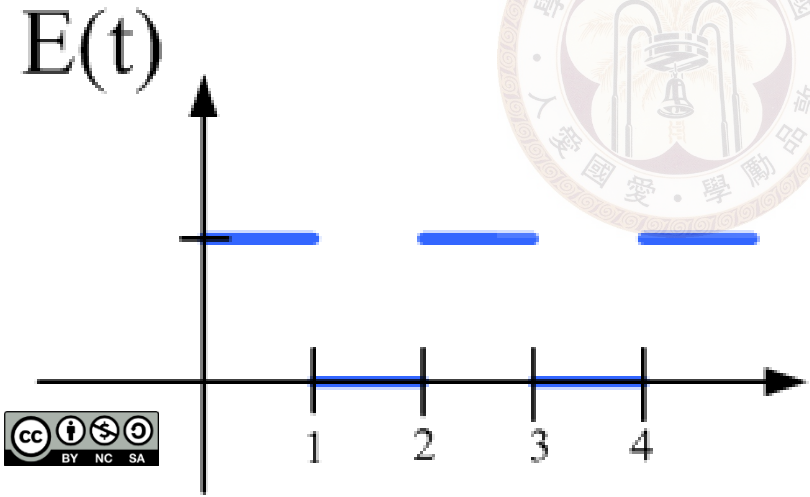
$$\Rightarrow L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

Example 7 (text page 288)

Square Wave (方波) 的例子

$$E(t)=\begin{cases}1 & \text{for } 0\le t<1\\ 0 & \text{for } 1\le t<2\end{cases}$$

$$E(t+2)=E(t)$$



$$L\{E(t)\} = \frac{1}{1-e^{-2s}}\left[\int_0^1 1\cdot e^{-st}dt + \int_1^2 0\cdot e^{-st}dt\right]$$

$$= \frac{1}{1-e^{-2s}}\frac{1-e^{-s}}{s} = \frac{1}{(1-e^{-s})(1+e^{-s})}\frac{1-e^{-s}}{s} = \frac{1}{s(1+e^{-s})}$$

Example 8 (text page 288)

$$L_1 \frac{d}{dt} i + Ri = E(t) \qquad i(0) = 0$$

$E(t)$ 為 page 476 之方波

$$sL_1 I(s) - L_1 i(0) + RI(s) = \frac{1}{s(1+e^{-s})}$$

$$I(s) = \frac{1/L_1}{(s+R/L_1)s(1+e^{-s})}$$

$$I(s)=\frac{1/L}{(s+R/L)s}\cdot\frac{1}{1+e^{-s}}=\left(\frac{1/R}{s}-\frac{1/R}{s+R/L}\right)\cdot\frac{1}{1+e^{-s}}$$

$$=\left(\frac{1/R}{s}-\frac{1/R}{s+R/L}\right)\cdot\left(1-e^{-s}+e^{-2s}-e^{-3s}+\cdots\cdots\right)$$

長除法 $\dfrac{1}{1+x}=1-x^1+x^2-x^3+\cdots\cdots$

$$I(s)=\frac{1}{R}\left(\frac{1}{s}-\frac{e^{-s}}{s}+\frac{e^{-2s}}{s}-\frac{e^{-3s}}{s}+\cdots\cdots\right)$$

$$-\frac{1}{R}\left(\frac{1}{s+R/L}-\frac{e^{-s}}{s+R/L}+\frac{e^{-2s}}{s+R/L}-\frac{e^{-3s}}{s+R/L}+\cdots\cdots\right)$$

$$L^{-1}\left\{\frac{1}{s}\right\}=1 \qquad L^{-1}\left\{\frac{e^{-ks}}{s}\right\}=u(t-k)$$

$k = 0, 1, 2, 3, \ldots..$

$$\frac{e^{-ks}}{s+R/L} = e^{-ks} \times \frac{1}{s+R/L}$$

先使用 $L^{-1}\left\{\frac{1}{s+R/L}\right\} = e^{-\frac{R}{L}t}$ 的公式

再算出 $L^{-1}\left\{\frac{e^{-ks}}{s+R/L}\right\} = e^{-\frac{R}{L}(t-k)}u(t-k)$

註：雖然也可以用 $\frac{e^{-ks}}{s+R/L} = \left.\frac{e^{-k(s-R/L)}}{s}\right|_{s\to s+R/L}$ 來算

但是較麻煩且容易出錯

7-4-6 Section 7.4 要注意的地方

(1) 注意代公式的順序 (例：Page 480 例子)

(2) 熟悉 convolution

(3) 變成積分時，別忘了加上 initial value

如
$$\frac{Q(t)}{C}=\frac{Q(0)}{C}+\frac{1}{C}\int_0^t i(\tau)\,d\tau$$

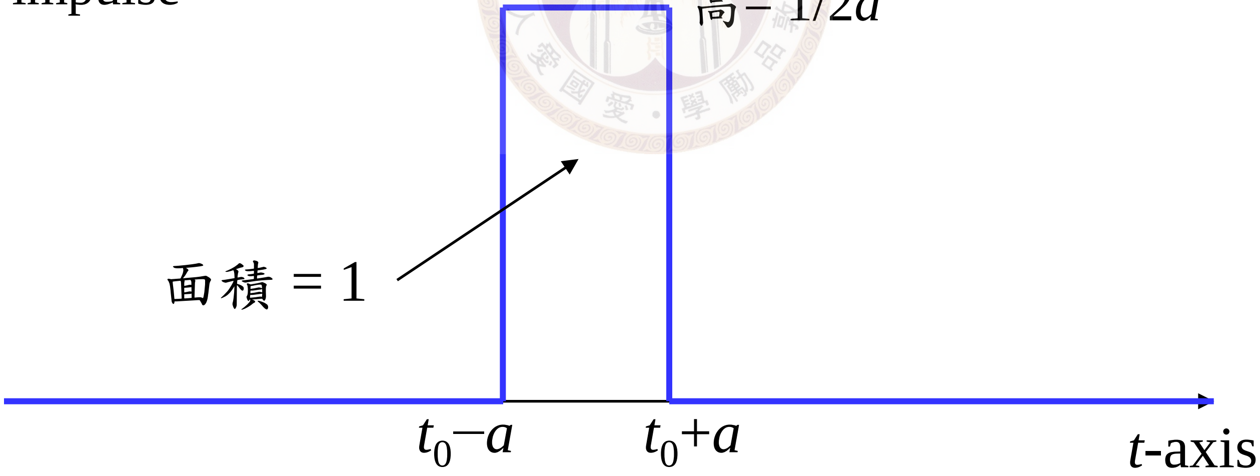
(4) 一定要記熟幾個重要的 properties (7 大性質)

Section 7.5 The Dirac Delta Function

7-5-1 Unit Impulse

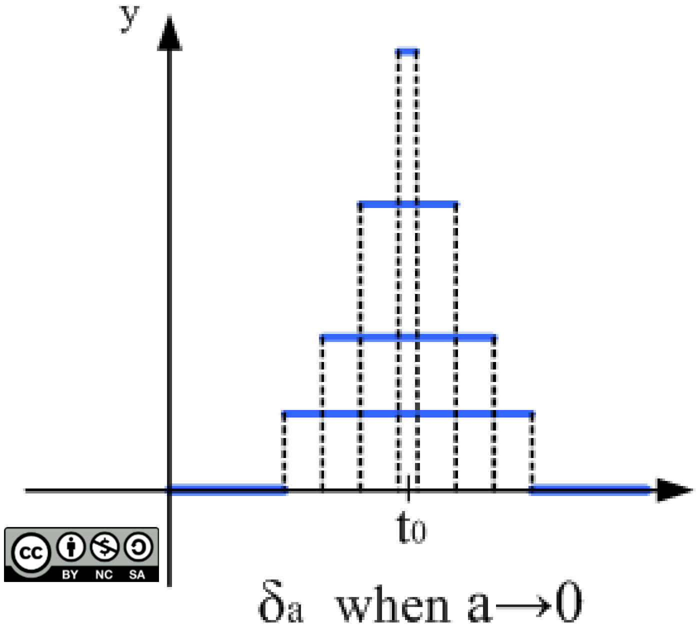
$$\delta_a(t-t_0)=\begin{cases}0 & \text{for } t<t_0-a \quad \text{or } t>t_0+a \\ \dfrac{1}{2a} & \text{for } t_0-a \le t \le t_0+a\end{cases}$$

稱作 unit impulse



7-5-2 Dirac Delta Function

$$\delta(t-t_0)=\lim_{a\to 0}\delta_a(t-t_0)$$



$$\delta_a(t-t_0)=\begin{cases}\infty & \text{for } t=t_0\\ 0 & \text{for } t\neq t_0\end{cases}$$

Fig. 7-5-2

7-5-3 Properties of the Dirac Delta Function

(1) Integration $$\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1$$

(2) Sifting $$\int_p^q f(t)\delta(t-t_0)\,dt = f(t_0)$$ when $t_0 \in [p, q]$

Proof: $$\int_p^q f(t)\delta(t-t_0)\,dt = \lim_{a\to 0}\int_p^q f(t)\delta_a(t-t_0)\,dt$$

$$\cong f(t_0)\lim_{a\to 0}\int_p^q \delta_a(t-t_0)\,d \; t = f(t_0)$$

當 a 很小的時候，$f(t) = f(t_0)$ for $t_0-a \le t \le t_0+a$

(3) Laplace transform of $\delta(t-t_0)$

$$L\left\{\delta\left(t-t_0\right)\right\}=e^{-t_0 s} \qquad \text{when } t_0>0$$

Proof: $\int_0^{\infty} e^{-st}\delta(t-t_0)dt$

(4) Relation with the unit step function

$$\int_{-\infty}^{t}\delta\left(\tau-t_0\right)d\tau=u\left(t-t_0\right)$$

$$\frac{d}{dt}u\left(t-t_0\right)=\delta\left(\tau-t_0\right)$$

7-5-4 Example

Example 1(a) (text page 293)

$$y'' + y = 4\delta(t - 2\pi) \qquad y(0) = 1 \qquad y'(0) = 0$$

$$s^2 Y(s) - s + Y(s) = 4e^{-2\pi s}$$

$$Y(s) = \frac{s}{s^2+1} + 4\frac{e^{-2\pi s}}{s^2+1} \qquad L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$$

$$y(t) = \cos t + 4\sin(t - 2\pi)u(t - 2\pi)$$

$$= \cos t + 4\sin t \cdot u(t - 2\pi)$$

$$y(t) = \begin{cases} \cos t & 0 \le t < 2\pi \\ \cos t + 4\sin t & t \ge 2\pi \end{cases}$$

7-5-5 幾個名詞

$$P(s)Y(s)-Q(s)=G(s) \longrightarrow Y(s)=W(s)Q(s)+W(s)G(s)$$

where $W(s)=\dfrac{1}{P(s)}$

(1) $w(t) = L^{-1}\{W(s)\}$ 稱作 weight function 或 impulse response

Note: When $Q(s) = 0$ (no initial condition) and $G(s) = 1$ ($g(t) = \delta(t)$),

$Y(s) = W(s)$, $y(t) = w(t)$.

(2) 許多文獻把 Dirac delta function $\delta(t - t_0)$ 亦稱作 delta function ，

impulse function，或 unit impulse function

7-5-6 本節要注意的地方

(1) Dirac delta function 不滿足 Theorem 7.1.3

(2) 幾個定理記熟，本節即可應付自如

Section 7-6 Systems of Linear Differential Equations

Chapter 7 的應用題

比較：類似的問題，也曾經在 Section 4-8 出現過

7-6-1 雙彈簧的例子

7-6-2 電路學的例子

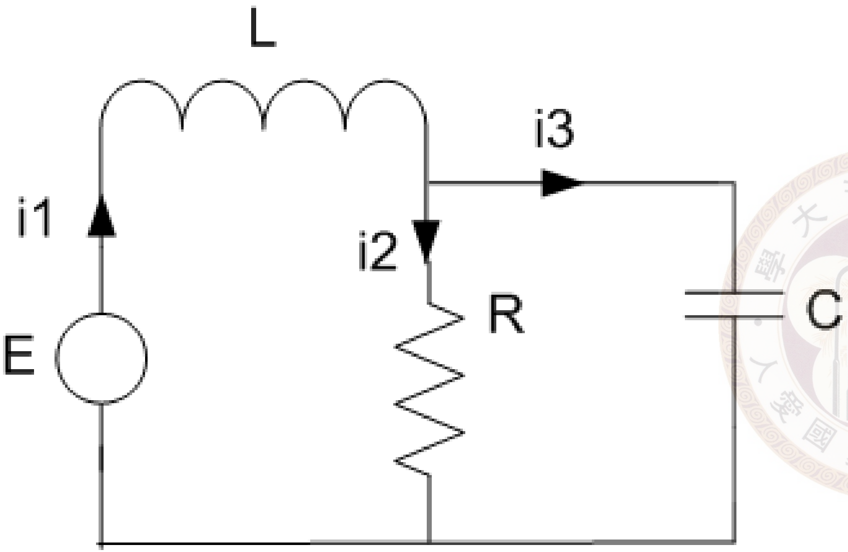


Fig. 3-3-4

Fig. 7-6-2

$$\begin{cases} L\dfrac{di_1(t)}{dt}+i_2R_2=E(t) \\ \dfrac{q_3}{C}=i_2R \\ i_1=i_2+i_3 \end{cases}$$

(由第2, 3 個式子)

$$i_1=i_2+\frac{d}{dt}q_3=i_2+\frac{d}{dt}RCi_2$$

$$\begin{cases} L\dfrac{di_1(t)}{dt}+i_2R_2=E(t) \\ RC\dfrac{d}{dt}i_2+i_2-i_1=0 \end{cases}$$

$$\begin{cases} L\dfrac{di_1(t)}{dt}+i_2R_2=E(t) \\ RC\dfrac{d}{dt}i_2+i_2-i_1=0 \end{cases}$$

Example 2 (text page 297)

$E(t) = 60$ V, $L = 1$ h, $R = 50\ \Omega$, $C = 10^{-4}$ f, $i_1(t) = i_2(t) = 0$

$$\begin{cases} \dfrac{di_1(t)}{dt}+50i_2=60 \\ 0.005\dfrac{d}{dt}i_2+i_2-i_1=0 \end{cases} \longrightarrow \begin{cases} sI_1(s)+50I_2(s)=\dfrac{60}{s} & \text{........... (式1)} \\ -I_1(s)+(0.005s+1)I_2(s)=0 & \text{...... (式2)} \end{cases}$$

(式1) × 1 + (式2) × s

$$(0.005s^2+s+50)I_2(s)=\frac{60}{s} \qquad (s^2+200s+10000)I_2(s)=\frac{12000}{s}$$

$$I_2(s)=\frac{12000}{s(s+100)^2}=\frac{6/5}{s}-\frac{120}{(s+100)^2}-\frac{6/5}{s+100}$$

複習：分子如何算出？

$$\boxed{i_2(t)=\frac{6}{5}-120te^{-100t}-\frac{6}{5}e^{-100t}}$$

將 $I_2(s)=\frac{6/5}{s}-\frac{120}{(s+100)^2}-\frac{6/5}{s+100}$ 代入式 (1)

$$sI_1(s)=\frac{60}{s}-\frac{60}{s}+\frac{6000}{(s+100)^2}+\frac{60}{s+100}$$

$$I_1(s)=\frac{6000}{s(s+100)^2}+\frac{60}{s(s+100)}$$

$$I_1(s)=\frac{6000}{s(s+100)^2}+\frac{60}{s(s+100)}=\frac{a}{s}+\frac{b+c(s+100)}{(s+100)^2}$$

$$a=\left.\frac{6000}{(s+100)^2}+\frac{60}{(s+100)}\right|_{s=0}=\frac{6}{5}$$

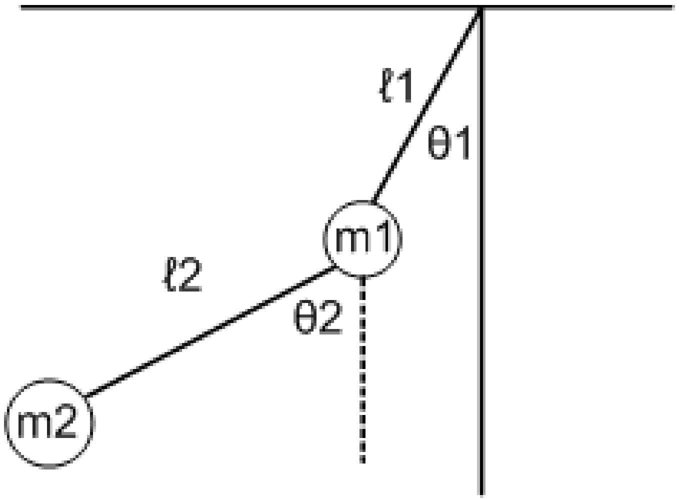
$$b=\left.\frac{6000}{s}+\frac{60(s+100)}{s}\right|_{s=-100}=-60$$

$$c=\left.\frac{d}{ds}\left(\frac{6000}{s}+\frac{60(s+100)}{s}\right)\right|_{s=-100}=\left.-\frac{6000}{s^2}-\frac{6000}{s^2}\right|_{s=-100}=-\frac{6}{5}$$

$$I_1(s)=\frac{6/5}{s}-\frac{60}{(s+100)^2}-\frac{6/5}{s+100}$$

$$\boxed{i_1(t)=\frac{6}{5}-60te^{-100t}-\frac{6}{5}e^{-100t}}$$

7-6-3 Double Pendulum (雙單擺) 的例子



$$(m_1+m_2)l_1^2\theta_1''+m_2l_1l_2\theta_2''+(m_1+m_2)l_1g\theta_1=0$$

$$m_2l_2^2\theta_2''+m_2l_1l_2\theta_1''+m_2l_2g\theta_2=0$$

$$(m_1 + m_2) l_1^2 \theta_1'' + m_2 l_1 l_2 \theta_2'' + (m_1 + m_2) l_1 g \theta_1 = 0$$

$$m_2 l_2^2 \theta_2'' + m_2 l_1 l_2 \theta_1'' + m_2 l_2 g \theta_2 = 0$$

Example 3 (text page 298)

$m_1 = 3,\ m_2 = 1,\ l_1 = 1_2 = 16,\ \theta_1(0) = 1,\ \theta_2(0) = -1,$

$\theta'_1(0) = 0,\ \theta'_2(0) = 0$

$$1024\theta_1'' + 256\theta_2'' + 64g\theta_1 = 0 \longrightarrow 16\theta_1'' + 4\theta_2'' + g\theta_1 = 0$$

$$256\theta_1'' + 256\theta_2'' + 16g\theta_2 = 0 \qquad 16\theta_1'' + 16\theta_2'' + g\theta_2 = 0$$

Laplace

$$16s^2\Phi_1(s) + 4s^2\Phi_2(s) + g\Phi_1(s) = 16s - 4s = 12s$$

$$16s^2\Phi_1(s) + 16s^2\Phi_2(s) + g\Phi_2(s) = 16s - 16s = 0$$

$$(16s^2+g)\Phi_1(s)+4s^2\Phi_2(s)=12s \quad \text{..............} (\text{式}1)$$

$$16s^2\Phi_1(s)+(16s^2+g)\Phi_2(s)=0 \quad \text{..............} (\text{式}2)$$

$$(\text{式}1)\times(16s^2+g)-(\text{式}2)\times 4s^2$$

$$[(16s^2+g)^2-64s^4]\Phi_1(s)=12s(16s^2+g)$$

$$[192s^4+32s^2g+g^2]\Phi_1(s)=12s(16s^2+g)$$

$$\Phi_1(s)=\frac{12s(16s^2+g)}{192s^4+32s^2g+g^2}=\frac{192s^3+12gs}{(24s^2+g)(8s^2+g)}=\frac{as+b}{24s^2+g}+\frac{cs+d}{8s^2+g}$$

$$\underline{(8a+24c)s^3}+(8b+24d)s^2+\underline{(a+c)gs}+bg+dg=192s^3+12gs$$

$$\begin{cases}8a+24c=192\\ a+c=12\end{cases}\Rightarrow\begin{cases}a=6\\ c=6\end{cases}\qquad\begin{cases}8b+24b=0\\ b+d=0\end{cases}\Rightarrow\begin{cases}b=0\\ d=0\end{cases}$$

$$\Phi_1(s)=\frac{6s}{24s^2+g}+\frac{6s}{8s^2+g}=\frac{6}{24}\frac{s}{s^2+(g/24)}+\frac{6}{8}\frac{s}{s^2+(g/8)}$$

$$\boxed{\theta_1(t)=\frac{1}{4}\cos\left(\frac{\sqrt{g}}{2\sqrt{6}}t\right)+\frac{3}{4}\cos\left(\frac{\sqrt{g}}{2\sqrt{2}}t\right)}$$

將 $\Phi_1(s)=\dfrac{12s(16s^2+g)}{192s^4+32s^2g+g^2}$ 代入 (式2)

$$\Phi_2(s)=-\frac{16s^2}{16s^2+g}\Phi_1(s)=-\frac{192s^3}{(24s^2+g)(8s^2+g)}=\frac{as+b}{24s^2+g}+\frac{cs+d}{8s^2+g}$$

直接用之前的式子

$$\begin{cases}8a+24c=-192\\ a+c=0\end{cases}\Rightarrow\begin{cases}a=12\\ c=-12\end{cases}\qquad\begin{cases}8b+24b=0\\ b+d=0\end{cases}\Rightarrow\begin{cases}b=0\\ d=0\end{cases}$$

$$\Phi_2(s) = \frac{12s}{24s^2+g} - \frac{12s}{8s^2+g} = \frac{12}{24}\frac{s}{s^2+(g/24)} - \frac{12}{8}\frac{s}{s^2+(g/8)}$$

$$\boxed{\theta_2(t) = \frac{1}{2}\cos\left(\frac{\sqrt{g}}{2\sqrt{6}}t\right) - \frac{3}{2}\cos\left(\frac{\sqrt{g}}{2\sqrt{2}}t\right)}$$

分式分解快速驗算技巧

將 $s = 0, s = 1,$ 或其他的值代入，看等號是否成立

7-6-4 本節需要注意的地方

(1) 正負號勿寫錯

(2) 要熟悉聯立方程式的變數消去法

(3) 多學習，甚至多「研發」簡化計算的技巧

Review of Chapter 7

(1) Laplace transform 定義

$$F(s)=\int_0^{\infty} e^{-st} f(t)dt$$

Inverse Laplace transform

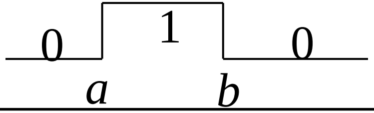
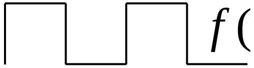
If $f(t) \xrightarrow{Laplace} F(s)$ and $f(t)$ is piecewise continuous of exponential order

then $F(s) \xrightarrow{inverse\ \ Laplace} f(t)$

(2) 7 大transform pairs

(看講義 page 413)

transform pairs 補充

$f(t)$	$F(s)$
$t\sin(kt)$	$\frac{2ks}{(s^2+k^2)^2}$
$t\cos(kt)$	$\frac{s^2-k^2}{(s^2+k^2)^2}$
$t\sinh(kt)$	$\frac{-2ks}{(s^2-k^2)^2}$
$t\cosh(kt)$	$\frac{s^2+k^2}{(s^2-k^2)^2}$
$u(t-a)$	e^{-as}/s
$\delta(t)$	1
	$\frac{e^{-as}-e^{-bs}}{s}$
 $f(t) = f(t+2a)$	$\frac{1}{s(1+e^{-as})}$

(3) 7 大 properties

input	Laplace transform
(1) Differentiation (Sec 7-2) $f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-\cdots\cdots$ $-sf^{(n-2)}(0)-f^{(n-1)}(0)$
(2) Multiplication by *t* (Sec7-4) $t^nf(t)$	$(-1)^n\dfrac{d^n}{ds^n}F(s)$
(3) Integration (Sec 7-4) $\int_0^t f(\tau)d\tau$	$\dfrac{F(s)}{s}$

(續)

input	Laplace transform
(4) Multiplication by exp (Sec7-3) $e^{at} f(t)$	$F(s-a)$
(5.1) Translation (Sec 7-3) $f(t-a)u(t-a)$	$e^{-as}F(s)$
(5.2) Translation (Sec 7-3) $g(t)u(t-a)$	$e^{-as}L\{g(t+a)\}$
(6) Convolution (Sec 7-4) $\int_0^t f(\tau)g(t-\tau)d\tau$	$F(s)G(s)$
(7) Periodic Input (Sec 7-4) $f(t) = f(t + T)$	$\frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)dt$

Properties 補充

input	Laplace transform
Scaling $f(t / a)$	$aF(as)$
Multiple Integrations $\int_0^t \int_0^{\tau_n} \cdots \int_0^{\tau_3} \int_0^{\tau_2} f\left(\tau_1\right) d\tau_1 d\tau_2 \cdots d\tau_{n-1} d\tau_n$	$\dfrac{F(s)}{s^n}$
Integration for s $f(t) / t$	$\int_s^\infty F\left(s_1\right) ds_1$

(4) 簡化運算的方法

分式分解 (see pages 432-437)

Initial conditions (see pages 445, 446)

(5) Delta function 的四大性質

Pages 484, 485

(6) General solutions

Laplace transform 的 general solution，可以用 initial conditions 來表示。

例子： $f''(t)-4f(t)=0$ 用Sec. 4-3 的方法解出 $f(t)=c_1e^{2t}+c_2e^{-2t}$

用 Laplace transform ：

$$s^2F(s)-sf(0)-f'(0)-4F(s)=0$$

$$F(s)=\frac{sf(0)+f'(0)}{s^2-4}=f(0)\frac{s}{s^2-4}+\frac{f'(0)}{2}\frac{2}{s^2-4}$$

$$f(t)=f(0)\cosh 2t+\frac{f'(0)}{2}\sinh 2t$$

和 Section 4-3 的解互相比較

$$f(t)=\frac{2f(0)+f'(0)}{4}e^{2t}+\frac{2f(0)-f'(0)}{4}e^{-2t}$$

$$f(0)=c_1+c_2$$

$$f'(0)=2c_1-2c_2$$

將 $f(0)=c_1+c_2$ $f'(0)=2c_1-2c_2$ 代入

$$f(t)=c_1e^{2t}+c_2e^{-2t}$$

Exercise for Practice

Sec. 7-1: 5, 8, 9, 18, 32, 33, 38, 41, 46, 48

Sec. 7-2: 11, 20, 23, 27, 30, 36, 37, 41, 42, 43

Sec. 7-3: 10, 16, 19, 20, 24, 34, 44, 56, 62, 70, 74, 83

Sec. 7-4: 8, 13, 25, 28, 30, 43, 52, 53, 54, 59, 61

Sec. 7-5: 5, 8, 11, 12, 15

Sec. 7-6: 8, 11, 12, 14, 15

Review 7: 12, 24, 25, 29, 36, 37, 40, 41, 42

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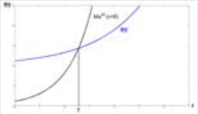
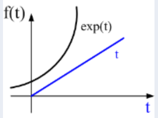
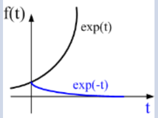
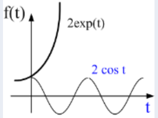
418		CC BY NC SA	國立臺灣大學 電信工程所 丁建均 教授
419		CC BY NC SA	國立臺灣大學 電信工程所 丁建均 教授
419		CC BY NC SA	國立臺灣大學 電信工程所 丁建均 教授
419		CC BY NC SA	國立臺灣大學 電信工程所 丁建均 教授

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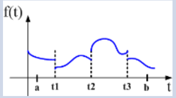
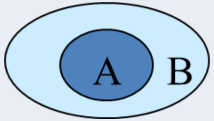
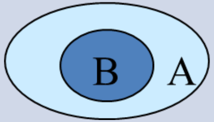
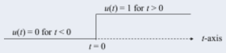
421		cc BY NC SA	國立臺灣大學 電信工程所 丁建均 教授
426		cc BY NC SA	國立臺灣大學 電信工程所 丁建均 教授
426		cc BY NC SA	國立臺灣大學 電信工程所 丁建均 教授
455		cc BY NC SA	國立臺灣大學 電信工程所 丁建均 教授

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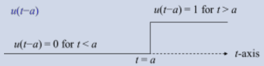
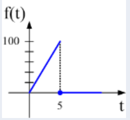
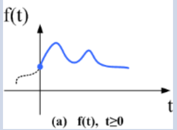
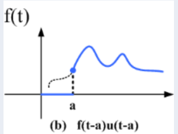
455		CC BY NC SA	國立臺灣大學 電信工程所 丁建均 教授
456		CC BY NC SA	國立臺灣大學 電信工程所 丁建均 教授
458		CC BY NC SA	國立臺灣大學 電信工程所 丁建均 教授
458		CC BY NC SA	國立臺灣大學 電信工程所 丁建均 教授

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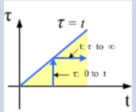
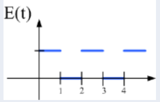
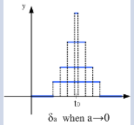
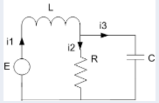
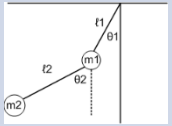
470		cc BY NC SA	國立臺灣大學 電信工程所 丁建均 教授
476		cc BY NC SA	國立臺灣大學 電信工程所 丁建均 教授
483		cc BY NC SA	國立臺灣大學 電信工程所 丁建均 教授
490		cc BY NC SA	國立臺灣大學 電信工程所 丁建均 教授
494		cc BY NC SA	國立臺灣大學 電信工程所 丁建均 教授