# 工程數學--微分方程

# Differential Equations (DE)

授課者：丁建均

教學網頁：http://djj.ee.ntu.edu.tw/DE.htm

期末考的範圍遠遠多於期中考，要了解的定理和觀念也非常多

希望同學們能及早準備

# Chapter 6 Series Solutions of Linear Equations

假設 DE 的 solutions 為polynomial 的型態

(和 Cauchy-Euler Method 以及 Taylor Series 的概念相近)

$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

被稱作為 power series centered at $x_0$

- $x_0$ is a non-singular point (Sec. 6-1)
- $x_0$ is a singular point
  - regular singular point (Sec. 6-2)
  - irregular singular point

(Sec. 6-3): examples

# Section 6-1 Solutions about Ordinary Points

假設解為 $\sum_{n=0}^{\infty} c_n (x - x_0)^n$

## 6-1-1 方法適用情形

(1) Linear

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots\cdots + a_1(x)y' + a_0(x)y = g(x)$$

(2) $x_0$ is not a singular point

(3) It is better that $a_0(x)$, $a_1(x)$, …., $a_n(x)$, $g(x)$ are all polynomials.

(or expressed as a Taylor series)

# 6-1-2 解法流程

Step 1 將 $y(x)=\sum_{n=0}^{\infty}c_n(x-x_0)^n$ 代入 ($x_0$ 必需為 ordinary point)

不是 singular point 的
即為 ordinary point

Step 2 對齊 (一律變成 $x^k$)

Step 3 合併

Step 4 比較係數，將 $c_n$ 之間的關係找出來

Step 5 Obtained independent solutions and general solution

# 6-1-3 例子

Example 3 (text page 224)

$$y'' + xy = 0$$

Set $y(x) = \sum_{n=0}^{\infty} c_n x^n$ since $P(x) = 0$ and $Q(x) = x$ are analytic at 0

Step 1

$$y'' + xy = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + x\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

set $k = n - 2$ set $k = n + 1$

Step 2 對齊

$$\sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k + \sum_{k=1}^{\infty} c_{k-1}x^k = 0$$

$$\sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k + \sum_{k=1}^{\infty} c_{k-1}x^k = 0$$

Step 3 $$2c_2 + \sum_{k=1}^{\infty}[c_{k+2}(k+2)(k+1) + c_{k-1}]x^k = 0$$

Step 4 $$2c_2 = 0$$ $$c_{k+2}(k+2)(k+1) + c_{k-1} = 0$$

$$c_2 = 0$$

或 $$c_{k+2} = \frac{-c_{k-1}}{(k+2)(k+1)}$$

recurrence relation

$c_0, c_1$ 給定之後

$k = 1 \quad c_3 = -\frac{c_0}{2\cdot 3}$

$k = 2 \quad c_4 = -\frac{c_1}{3\cdot 4}$

$k = 3 \quad c_5 = -\frac{c_2}{4\cdot 5} = 0$

$$c_{k+2} = \frac{-c_{k-1}}{(k+2)(k+1)}$$

$k = 4 \quad c_6 = \frac{-c_3}{5\cdot 6} = \frac{c_0}{2\cdot 3\cdot 5\cdot 6}$

$k = 5 \quad c_7 = -\frac{c_4}{6\cdot 7} = \frac{c_1}{3\cdot 4\cdot 6\cdot 7}$

$k = 6 \quad c_8 = -\frac{c_5}{7\cdot 8} = 0$

$k = 7 \quad c_9 = \frac{-c_6}{8\cdot 9} = -\frac{c_0}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}$

$k = 8 \quad c_{10} = \frac{-c_7}{9\cdot 10} = -\frac{c_1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 1}$

$k = 9 \quad c_{11} = \frac{-c_8}{10\cdot 11} = 0$

⋮

以此類推，所有的 $c_n$ 的值都可以算出來(以 $c_0$ 或 $c_1$ 表示)

Step 5

$$y(x)=\sum_{n=0}^{\infty}c_nx^n=c_0\left[1-\frac{x^3}{2\cdot3}+\frac{x^6}{2\cdot3\cdot5\cdot6}-\frac{x^9}{2\cdot3\cdot5\cdot6\cdot8\cdot9}+\cdots\cdots\right] \quad y_1$$

$$+c_1\left[x-\frac{x^4}{3\cdot4}+\frac{x^7}{3\cdot4\cdot6\cdot7}-\frac{x^{10}}{3\cdot4\cdot6\cdot7\cdot9\cdot10}+\cdots\cdots\right] \quad y_2$$

$$y(x)=c_0y_1(x)+c_1y_2(x)$$

$$y_1(x)=1+\sum_{k=1}^{\infty}\frac{(-1)^k}{2\cdot3\cdots(3k-1)(3k)}x^{3k}$$

$$y_2(x)=x+\sum_{k=1}^{\infty}\frac{(-1)^k}{3\cdot4\cdots(3k)(3k+1)}x^{3k+1}$$

Example 4 (text page 226)

$$(x^2+1)y'' + xy' - y = 0 \qquad y(x) = \sum_{n=0}^{\infty} c_n x^n \quad \text{(analytic at } x = 0\text{)}$$

Radius of convergence?

Step 1

$$(x^2+1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} c_n x^n = 0$$

$k = n$ $\quad$ $k = n - 2$ $\quad$ $k = n$ $\quad$ $k = n$

Step 2

$$\sum_{k=2}^{\infty} k(k-1)c_k x^k + \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2} x^k + \sum_{k=1}^{\infty} k c_k x^k - \sum_{k=0}^{\infty} c_k x^k = 0$$

$$\sum_{k=2}^{\infty} k(k-1)c_k x^k + \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=1}^{\infty} kc_k x^k - \sum_{k=0}^{\infty} c_k x^k = 0$$

Step 3

$$\underbrace{2c_2 - c_0}_{k=0} + \underbrace{(6c_3 + c_1 - c_1)x}_{k=1}$$

$$+\sum_{k=2}^{\infty}[k(k-1)c_k + (k+2)(k+1)c_{k+2} + kc_k - c_k]x^k = 0$$

$$2c_2 - c_0 + 6c_3 x + \sum_{k=2}^{\infty}[(k+1)(k-1)c_k + (k+2)(k+1)c_{k+2}]x^k = 0$$

Step 4

$$\boxed{2c_2 - c_0 = 0} \qquad \boxed{6c_3 = 0} \qquad \boxed{c_{k+2} = -\frac{k-1}{k+2}c_k}$$

$2c_2 - c_0 = 0$ $6c_3 = 0$ $c_{k+2} = \frac{1-k}{k+2} c_k$

$c_2 = c_0/2$ $c_3 = 0$

$c_0, c_1$ 給定之後

$$c_2 = c_0/2$$

$$c_3 = 0$$

$$c_4 = -\frac{1}{4}c_2 = -\frac{1}{2\cdot 4}c_0 = -\frac{1}{2^2\cdot 2!}c_0$$

$$c_5 = -\frac{2}{5}c_3 = 0$$

$$c_6 = -\frac{3}{6}c_4 = \frac{3}{2\cdot 4\cdot 6}c_0 = \frac{3}{2^3\cdot 3!}c_0$$

$$c_7 = -\frac{4}{7}c_5 = 0$$

$$c_8 = -\frac{5}{8}c_6 = -\frac{3\cdot 5}{2\cdot 4\cdot 6\cdot 8}c_0 = -\frac{3\cdot 5}{2^4\cdot 4!}c_0$$

$$c_9 = -\frac{6}{9}c_7 = 0$$

:

:

Step 5

$$y(x)=\sum_{n=0}^{\infty}c_n x^n = c_0\underbrace{\left[1+\frac{x^2}{2}-\frac{x^4}{2^2\cdot 2!}+\frac{3x^6}{2^3\cdot 3!}-\frac{3\cdot 5x^8}{2^4\cdot 4!}+\frac{3\cdot 5\cdot 7x^{10}}{2^5\cdot 5!}\cdots\cdots\right]}_{y_1}+c_1\underbrace{x}_{y_2}$$

$$y(x)=c_0 y_1(x)+c_1 y_2(x)$$

$$y_1(x)=1+\frac{1}{2}x^2+\sum_{n=2}^{\infty}(-1)^{n-1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n n!}x^{2n}$$

$|x| < 1$ (Why?)

$$y_2(x)=x$$

# Example 6 (text page 228)

$$y'' + (\cos x)y = 0$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots\cdots\cdots$$

$$y_1 = 1 - \frac{1}{2}x^2 + \frac{1}{12}x^4 - \cdots\cdots\cdots \qquad y_2 = x - \frac{1}{6}x^3 + \frac{1}{30}x^5 - \cdots\cdots\cdots$$

# 6-1-4 重要定義

1. Convergence: $\lim_{N\to\infty}\sum_{n=0}^{N} c_n (x-x_0)^n$ exists

測試方法：Ratio test (test for convergence)

$$\lim_{n\to\infty}\left|\frac{c_{n+1}(x-x_0)^{n+1}}{c_n(x-x_0)^n}\right| = L$$

$L<1$: converge　$L>1$: diverge

$L=1$: 不一定

2. Radius of Convergence $R$

$L<1$ if $|x-x_0|<R$　　$L>1$ if $|x-x_0|>R$

3. Analytic at $x_0$: If the radius of convergence is nonzero

簡單的判斷 $f(x)$ 在 $x_0$ 是否為analytic 方法

(1) $f(x_0)$ should be neither $\infty$ nor $-\infty$

(2) $f^{(m)}(x_0)$ should be neither $\infty$ nor $-\infty$

$m = 1, 2, 3, \ldots\ldots\ldots$

- For the 2nd order linear DE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \longrightarrow y'' + P(x)y' + Q(x)y = 0$$

**Definition 6.1**

$x_0$ is an ordinary point of the 2nd order linear DE if both $P(x)$ and $Q(x)$ are **analytic** at $x_0$

Otherwise, $x_0$ is a singular point.

**Theorem 6.1**

If $x_0$ is an ordinary point of the 2nd order linear DE, then we can find two linearly independent solutions in the form of a power series centered at $x_0$, i.e.,

$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

- For the $k^{\text{th}}$ order linear DE

$$a_k(x)y^{(k)}+a_{k-1}(x)y^{(k-1)}+\cdots\cdots+a_1(x)y'+a_0(x)y=0$$

$$\longrightarrow\quad y^{(k)}+P_{k-1}(x)y^{(k-1)}+\cdots\cdots+P_1(x)y'+P_0(x)y=0$$

$$P_m(x)=\frac{a_m(x)}{a_n(x)}$$

**Extension of Definition 6.1**

$x_0$ is an ordinary point of the $k^{\text{th}}$ order linear DE if $P_0(x)$, $P_1(x)$, $P_2(x)$, ………. , $P_{k-1}(x)$, are **analytic** at $x_0$

Otherwise, $x_0$ is a singular point .

**Extension of Theorem 6.1**

If $x_0$ is an ordinary point of the $n^{\text{th}}$ order linear DE, then we can find $n$ linearly independent solutions in the form of a power series centered at $x_0$ , i.e.,

$$y(x)=\sum_{n=0}^{\infty}c_n(x-x_0)^n$$

## 6-1-5 思考

(1) 對於 nonhomogeneous 的情形.....

$$a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_1(x)y'(x)+a_0(x)y(x)=g(x)$$

(2) 這方法還可以用在什麼情形？

# 6-1-6 Interval of Convergence 的判斷方法

**判斷方法一**：找出 $\lim_{n\to\infty}\left|\frac{c_{n+1}(x-x_0)^{n+1}}{c_n(x-x_0)^n}\right|<1$ 的條件

**判斷方法二**

（較快速，但較不精準

找出的收斂的範圍有時會比實際的收斂範圍小）

$$|x-x_0|<R$$

其中 $R$ 是 $x_0$ 和最近的 singular point 的距離

Singular point can be a complex number , see the example in page 345

超過這個範圍未必不為 convergence

## 6-1-7 本節需注意的地方

(1) 要了解幾個重要定義： (a) convergence, (b) radius of convergence,

(c) analytic at $x_0$, (d) singular point, (e) ordinary point

(2) 複習一下某些 function 的 Taylor series (如後頁)

(3) Index 的地方計算要小心

(a) 先都化成 $x^k$ 再合併，(b) 頭幾項可能要獨立出來

(c) Index 對齊計算要小心

(4) $n^{th}$ order linear DE 要有 $n$ 個 linearly independent 解

(5) 有時要考慮 interval of convergence

附：Taylor series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots\cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\cdots\cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots\cdots\cdots$$

$$e^{2x} = ?$$

$$\cos 3x = ?$$

# Section 6-2 Solutions about Singular Points

$$y^{(n)} + P_{n-1}(x)\,y^{(n-1)} + \cdots\cdots + P_1(x)\,y' + P_0(x)\,y = 0$$

假設解為 $y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$

## 6-2-1 方法適用情形

(1) Linear

(2) $(x - x_0)P_{n-1}(x)$, $(x - x_0)^2 P_{n-2}(x)$, ………….. , $(x - x_0)^{n-1} P_1(x)$,

$(x - x_0)^n P_0(x)$ are **analytic** at $x_0$

(比較： Section 6-1 要求 $P_{n-1}(x)$, $P_{n-2}(x)$, ………….. ,

$P_1(x)$, $P_0(x)$ are **analytic** at $x_0$)

(3) It is better that $P_0(x)$, $P_1(x)$, …., $P_{n-1}(x)$ are all polynomials.

# 6-2-2 定義

## Singular Points 分成二種

- If $x_0$ is a singular point but $(x-x_0)P_{n-1}(x)$, $(x-x_0)^2P_{n-2}(x)$, ……….… , $(x-x_0)^{n-1}P_1(x)$, $(x-x_0)^nP_0(x)$ are analytic at $x_0$

$x_0$ : regular singular point

- If $(x-x_0)P_{n-1}(x)$, $(x-x_0)^2P_{n-2}(x)$, ……….… , $(x-x_0)^{n-1}P_1(x)$, $(x-x_0)^nP_0(x)$ are not analytic at $x_0$

$x_0$ : irregular singular point

Example 1 (text page 232)

$$(x^2-4)^2 y'' + 3(x-2)y' + 5y = 0$$

$$P(x) = \frac{3}{(x-2)(x+2)^2} \qquad Q(x) = \frac{5}{(x-2)^2(x+2)^2}$$

$x = 2$ is a point

$x = -2$ is a point

# 6-2-3 解法

**解法的關鍵：**

假設解為 $y(x)=\sum_{n=0}^{\infty} c_n (x-x_0)^{n+r}$

**Theorem 6.2 Frobenius' Theorem**

若 $x_0$ 是 linear DE 當中的一個 regular singular point

則這個 linear DE 至少有一個解是 $y(x)=\sum_{n=0}^{\infty} c_n (x-x_0)^{n+r}$ 的型態

Process

Step 1 將 $y(x)=\sum_{n=0}^{\infty}c_n(x-x_0)^{n+r}$ 代入

Step 2 Power 對齊

Step 3 合併

Step 4 算出 $r$

Step 5 比較係數，將 $c_n$ 之間的關係找出來

Step 6 將 Step 4 得出的 $r$ 代入 Step 5

得出所有的 independent solutions 及 general solution

Step 7 (見後頁)

## (Step 7)

當 (1) $r$ 有重根

(2) 或 $r$ 的根之間的差為整數，且從 Step 6 得出來的解不為 independent 時

用 $y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx$ 和長除法找出 $y_2(x)$

(參考 Section 6-2 的 Examples 4, 5)

當 $r$ 的根之間的差為整數，但從 Step 6 得出來的解為 independent 時，不需進行這個步驟

## 6-2-4 範例

Example 2 (text page 234)

$$3xy'' + y' - y = 0$$

Step 1 將 $y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$ 代入

$$3\sum_{n=0}^{\infty} c_n (n+r)(n+r-1)x^{n+r-1} + \sum_{n=0}^{\infty} c_n (n+r)x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

Step 2 Power 對齊

$k = n$ $\quad$ $k = n$ $\quad$ $n = k - 1$, $k = n + 1$

$$3\sum_{k=0}^{\infty} c_k (k+r)(k+r-1)x^{k+r-1} + \sum_{k=0}^{\infty} c_k (k+r)x^{k+r-1} - \sum_{k=1}^{\infty} c_{k-1} x^{k+r-1} = 0$$

## Step 3 合併

$$[3c_0 r(r-1)+c_0 r]x^{r-1}+\sum_{k=1}^{\infty}[3c_k(k+r)(k+r-1)+c_k(k+r)-c_{k-1}]x^{k+r-1}=0$$

## Step 4 算出 $r$

$$3r(r-1)+r=0 \qquad 3r^2-2r=0 \qquad r(3r-2)=0$$

$$\boxed{r=0 \quad or \quad 2/3}$$

## Step 5

$$3c_k(k+r)(k+r-1)+c_k(k+r)-c_{k-1}=0$$

$$c_k=\frac{1}{(k+r)(3k+3r-2)}c_{k-1}$$

$$c_k = \frac{1}{(k+r)(3k+3r-2)} c_{k-1}$$

## Step 6

當 $r = 0$ $\quad c_k = \frac{c_{k-1}}{k(3k-2)}$

$$c_1 = \frac{c_0}{1\cdot 1}$$

$$c_2 = \frac{c_1}{2\cdot 4} = \frac{c_0}{2!1\cdot 4}$$

$$c_3 = \frac{c_2}{3\cdot 7} = \frac{c_0}{3!1\cdot 4\cdot 7}$$

$$c_4 = \frac{c_3}{4\cdot 10} = \frac{c_0}{4!1\cdot 4\cdot 7\cdot 10}$$

$$\vdots$$

$$c_n = \frac{c_0}{n!1\cdot 4\cdot 7\cdots\cdots(3n-2)}$$

當 $r = 2/3$ $\quad c_k = \frac{1}{(3k+2)k} c_{k-1}$

$$c_1 = \frac{c_0}{5\cdot 1}$$

$$c_2 = \frac{c_1}{8\cdot 2} = \frac{c_0}{2!5\cdot 8}$$

$$c_3 = \frac{c_2}{11\cdot 3} = \frac{c_0}{3!5\cdot 8\cdot 11}$$

$$c_4 = \frac{c_3}{14\cdot 4} = \frac{c_0}{4!5\cdot 8\cdot 11\cdot 14}$$

$$\vdots$$

$$c_n = \frac{c_0}{n!5\cdot 8\cdot 11\cdots\cdots(3n+2)}$$

Solution of Example 2 (別忘了將最後的解寫出）

$$y = C_1 y_1 + C_2 y_2$$

$$y_1(x) = x^0 \left[ 1 + \sum_{n=1}^{\infty} \frac{1}{n! 1 \cdot 4 \cdot 7 \cdot \cdots \cdots (3n-2)} x^n \right]$$

$$y_2(x) = x^{2/3} \left[ 1 + \sum_{n=1}^{\infty} \frac{1}{n! 5 \cdot 8 \cdot 11 \cdots \cdots (3n+2)} x^n \right]$$

$$x \in (0, \infty)$$

(別忘了寫出 $x$ 的範圍)

Examples 4, 5 (text pages 237, 238)

$$xy'' + y = 0$$

Step 1 將 $y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$ 代入

$$\sum_{n=0}^{\infty} c_n (n+r)(n+r-1)x^{n+r-1} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

Step 2 對齊 $n = k$ $n = k-1$

$$\sum_{k=0}^{\infty} c_k (k+r)(k+r-1)x^{k+r-1} + \sum_{k=1}^{\infty} c_{k-1} x^{k+r-1} = 0$$

Step 3 合併

$$c_0 r(r-1)x^{r-1} + \sum_{k=1}^{\infty} [c_k (k+r)(k+r-1) + c_{k-1}]x^{k+r-1} = 0$$

Step 4 $r(r-1)=0$

$$r=0 \quad or \quad 1$$

Step 5 $$c_k=-\frac{c_{k-1}}{(k+r)(k+r-1)}$$

Step 6 當 $r=1$ $c_k=-\dfrac{c_{k-1}}{(k+1)k}$

$$c_n=(-1)^n\frac{c_0}{(n+1)!n!}$$

$$y_1(x)=x\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!(n+1)!}x^n\right]=x\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(n+1)!}x^n\right]$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(n+1)!}x^{n+1}$$

Step 6 當 $r = 0$ $c_k = -\dfrac{c_{k-1}}{(k-1)k}$

$k = 1$ 時不能算

此時，應該根據 Step 3，由

$$c_k(k+r)(k+r-1) + c_{k-1} = c_1 \cdot 0 + c_0 = 0$$

($k = 1, r = 0$ 代入)

$c_0$ 必需等於 0, $c_1$ 可為任意值

這地方容易犯錯，要小心

$$c_2 = -\frac{c_1}{1\cdot 2} \quad c_3 = -\frac{c_2}{2\cdot 3} = \frac{c_1}{1\cdot 2\cdot 2\cdot 3} \cdots c_n = \frac{(-1)^n c_1}{(n-1)!\cdot n!}$$

$$y_2(x) = x^0\left[x + \sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{(n-1)!n!}x^n\right] = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n-1)!n!}x^n = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!(m+1)!}x^{m+1}$$

$m = n - 1$

因為前頁算出來的 $y_2(x)$ 等於 $y_1(x)$，

只好另外用 “reduction of order” 的方法求解

$$y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx$$

$$y_2(x)=y_1(x)\int\frac{e^{-\int 0\,dx}}{y_1^2(x)}dx=y_1(x)\int\frac{dx}{[x-\frac{1}{2}x^2+\frac{1}{12}x^3-\frac{1}{144}x^4+\cdots\cdots]^2}$$

$$=y_1(x)\int\frac{dx}{[x^2-x^3+\frac{5}{12}x^4-\frac{7}{72}x^5+\cdots\cdots]}$$

long division

長除法

$$=y_1(x)\int\left[\frac{1}{x^2}+\frac{1}{x}+\frac{7}{12}+\frac{19}{72}x+\cdots\cdots\right]dx$$

$$=y_1(x)\left[-\frac{1}{x}+\ln x+\frac{7}{12}x+\frac{19}{144}x^2+\cdots\cdots\right]$$

思考：為何不是 ln|*x*| ?

$$=y_1(x)\ln x+y_1(x)\left[-\frac{1}{x}+\frac{7}{12}x+\frac{19}{144}x^2+\cdots\cdots\right]$$

$$=y_1(x)\ln x+\left[-1-\frac{1}{2}x+\frac{1}{2}x^2+\cdots\cdots\right]$$

# 6-2-5 多項式的長除法

計算 $\dfrac{1}{x^2 - x^3 + \frac{5}{12}x^4 - \frac{7}{12}x^5 + \cdots\cdots}$

$$
\begin{array}{r|llllll}
 & 1 & 1 & \frac{7}{12} & \frac{19}{72} & \cdots & \\
\hline
1 \quad -1 \quad \frac{5}{12} \quad -\frac{7}{72} \quad \cdots & 1 & & & & & \\
 & 1 & -1 & \frac{5}{12} & -\frac{7}{72} & \cdots & \\
\cline{2-7}
 & & 1 & -\frac{5}{12} & \frac{7}{72} & \cdots & \\
 & & 1 & -1 & \frac{5}{12} & -\frac{7}{72} & \cdots \\
\cline{2-7}
 & & & \frac{7}{12} & -\frac{23}{72} & \cdots & \cdots \\
 & & & \frac{7}{12} & -\frac{7}{12} & \frac{35}{144} & -\frac{49}{864} \quad \cdots \\
\cline{2-7}
 & & & & \frac{19}{72} & \cdots & \\
\end{array}
$$

# 6-2-6 Indicial Equation

2nd order case $y'' + P(x)y' + Q(x)y = 0$

If $x_0$ is a regular singular point

$$(x - x_0)^2 y'' + (x - x_0)p(x)y' + q(x)y = 0$$

where $p(x) = (x - x_0)P(x)$ $\quad q(x) = (x - x_0)^2 Q(x)$

由於 $p(x)$ 和 $q(x)$ 皆為 analytic

$$p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots\cdots$$

$$q(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)^2 + \cdots\cdots$$

$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}, \quad y'(x) = \sum_{n=0}^{\infty} c_n (n+r)(x - x_0)^{n+r-1}, \quad y''(x) = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1)(x - x_0)^{n+r-2},$$

將 $y(x), y'(x), y''(x), p(x), q(x)$ 代入

$$(x-x_0)^2 y'' + (x-x_0)p(x)y' + q(x)y = 0$$

$$\sum_{n=0}^{\infty} c_n(n+r)(n+r-1)(x-x_0)^{n+r}$$

$$+\left(a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots\cdots\cdots\right)\sum_{n=0}^{\infty} c_n(n+r)(x-x_0)^{n+r}$$

$$+\left(b_0 + b_1(x-x_0) + b_2(x-x_0)^2 + \cdots\cdots\cdots\right)\sum_{n=0}^{\infty} c_n(x-x_0)^{n+r} = 0$$

其中 $(x-x_0)^r$ 的 coefficient 為

$$c_0 r(r-1) + c_0 a_0 r + c_0 b_0$$

$$\boxed{r(r-1) + a_0 r + b_0 = 0} \longrightarrow$$ **indicial equation**

$$y(x)=\sum_{n=0}^{\infty}c_n(x-x_0)^{n+r}$$

當 linear DE 為 2nd order 時，$r$ 可以由 $r(r-1)+a_0r+b_0=0$ 求出

其中

$$a_0=p(x_0) \qquad p(x)=(x-x_0)P(x) \qquad y''+P(x)y'+Q(x)y=0$$

$$b_0=q(x_0) \qquad q(x)=(x-x_0)^2Q(x)$$

## For the 2nd order case

$$r(r-1)+a_0 r+b_0=0 \qquad \text{two roots: } r_1, r_2$$

(Case 1) $r_1 \neq r_2$ and $r_1, r_2$ are real, $r_2 - r_1 \neq$ integer

可以找出兩組 $y(x)=\sum_{n=0}^{\infty} c_n (x-x_0)^{n+r}$ 的解

(Case 2) $r_1 \neq r_2$ and $r_1, r_2$ are real, $r_2 - r_1 =$ integer

一個解是 $y_1(x)=\sum_{n=0}^{\infty} c_n (x-x_0)^{n+r_1}$

另一個解是 $y_2(x)=Cy_1(x)\ln x+\sum_{n=0}^{\infty} b_n (x-x_0)^{n+r_2}$

$C$ 有時等於 0 (和 case 1 相同)

有時不為 0

(Case 3) $r_1 = r_2$ 時

$$y_1(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r_1}$$

$$y_2(x) = C y_1(x) \ln x + \sum_{n=0}^{\infty} b_n (x - x_0)^{n+r_2}$$ $C$ 一定不為 0

或寫成 $$y_2(x) = y_1(x) \ln x + \sum_{n=0}^{\infty} \tilde{b}_n (x - x_0)^{n+r_2}$$ $\tilde{b}_n = b_n / C$

(Case 4) $r_1 \neq r_2$ and $r_1$, $r_2$ are complex

在此不予討論

# 6-2-7 Indicial Equation for Higher Order Case (補充)

當 linear DE 為 $n^{\text{th}}$ order 時

$$y^{(n)}+P_{n-1}(x)y^{(n-1)}+\cdots\cdots+P_1(x)y'+P_0(x)y=0$$

$$y(x)=\sum_{n=0}^{\infty}c_n(x-x_0)^{n+r}$$ 當中的 $r$ 可以由

$$\frac{r!}{(r-n)!}+a_{n-1,0}\frac{r!}{(r-n+1)!}+a_{n-2,0}\frac{r!}{(r-n+2)!}+\ldots\ldots+a_{1,0}\frac{r!}{(r-1)!}+a_{0,0}=0$$

求出

其中 $a_{k,0}=p_k(x_0)$, $p_k(x_0)=(x-x_0)^{n-k}P_k(x_0)$

$k=0,1,2,\ldots.,n-1$

# 6-2-8 本節需要注意的地方

(1) Index 對齊計算要小心

(建議可以向 power 較小的對齊，否則會出現負的 $k$)

(2) 若 $x = 0$ 為 regular singular point, 設 $x_0 = 0$ 即可

(3) 如果是 $c_k$ 和 $c_{k-1}$ (或 $c_{k-1}$ 和 $c_k$) 的 recursive relation

其實有時可以立刻將 $c_n$ 的式子觀察出來

(但是分母不可變為 0)

(4) 小心分母為 0 的情形 (如 page 370)

(5) 小心算出來的 $y_2(x)$ 和 $y_1(x)$ 相同的情形 (如 pages 370, 371)

(6)別忘了將最後的解寫出

最後的解易出錯的地方： 別忘了乘上 $x^r$

(7) Interval of solution 依然要考慮，

且 interval 不包括任何singular point，

即使是 regular singular point

(8) 複習長除法

# Section 6-3 Special Functions

Special cases of Sections 6-1 and 6-2

- **Bessel's equation of order *v***

$$x^2 y'' + xy' + (x^2 - v^2)y = 0$$

Solution: $c_1 J_v(x) + c_2 Y_v(x)$

$$J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+v+n)} \left(\frac{x}{2}\right)^{2n+v}$$

: 1st kind Bessel function

$$Y_v(x) = \frac{\cos v\pi J_v(x) - J_{-v}(x)}{\sin v\pi}$$

: 2nd kind Bessel function

- **Legendre's equation of order *n***

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0$$

One of the solution: Legendre polynomials (See page 399)

## 其他名詞

- Gamma function $\Gamma(x)=\int_0^{\infty} t^{x-1}e^{-t}dt$
- modified Bessel equation of order $v$

$$x^2y''+xy'-(x^2+v^2)y=0$$

解：$c_1I_v(x)+c_2K_v(x)$

- modified Bessel equation of the 1st kind $I_v(x)=i^{-v}J_v(ix)$
- modified Bessel equation of the 2nd kind $K_v(x)=\frac{\pi}{2}\frac{I_{-v}(x)-I_v(x)}{\sin v\pi}$
- Bessel 的另一種變型

$$x^2y''+(1-2a)xy'+(b^2c^2x^{2c}+a^2-p^2c^2)y=0$$

解 $y=x^a\left[c_1J_p(bx^c)+c_2Y_p(bx^c)\right]$

- spherical Bessel functions: $J_v(x)$, $v=\pm\frac{1}{2},\pm\frac{3}{2},\pm\frac{5}{2},\cdots\cdots$

# 6.3.1 Bessel's Equation

## 6.3.1.1 Solving for Bessel's equation of order *v*

$$x^2 y'' + xy' + (x^2 - v^2)y = 0$$

Steps 1~3 將 $y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$ 代入

經過一些計算 (詳見課本 page 241) 得出

$$c_0(r^2 - v^2)x^r + c_1((1+r)^2 - v^2)x^{r+1} + \sum_{k=2}^{\infty}[c_k((k+r)^2 - v^2) + c_{k-2}]x^{r+k} = 0$$

Step 4 $r^2 - v^2 = 0$ two roots: $v$ and $-v$

Step 5 $c_1\left((1+r)^2 - v^2\right) = 0$ $c_k = \dfrac{c_{k-2}}{v^2 - (k+r)^2}$

$c_1 = 0$

$$c_k = \frac{c_{k-2}}{v^2 - (k+r)^2}$$

Step 6 當 $r = v$ $\quad c_k = -\frac{c_{k-2}}{k(k+2v)}$ 當 $r = -v$ $\quad c_k = -\frac{c_{k-2}}{k(k-2v)}$

由於 $c_1 = 0$, $c_3 = c_5 = c_7 = c_9 = \ldots.. = 0$

$$c_{2n} = (-1)^n \frac{c_0}{2\cdot 4\cdot 6\cdots\cdots 2n\cdot(2+2v)(4+2v)(6+2v)\cdots\cdots(2n+2v)}$$

$$= \frac{(-1)^n c_0}{2^{2n} n!(1+v)(2+v)(3+v)\cdots\cdots(n+v)}$$

when $r = v$

$$c_{2n} = (-1)^n \frac{c_0}{2\cdot 4\cdot 6\cdots\cdots 2n\cdot(2-2v)(4-2v)(6-2v)\cdots\cdots(2n-2v)}$$

$$= \frac{(-1)^n c_0}{2^{2n} n!(1-v)(2-v)(3-v)\cdots\cdots(n-v)}$$

when $r = -v$

## 6.3.1.2 Gamma function: a generalization of $n!$

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

properties of Gamma function

(1) $\Gamma(n+1) = n!$ when $n$ is a positive integer

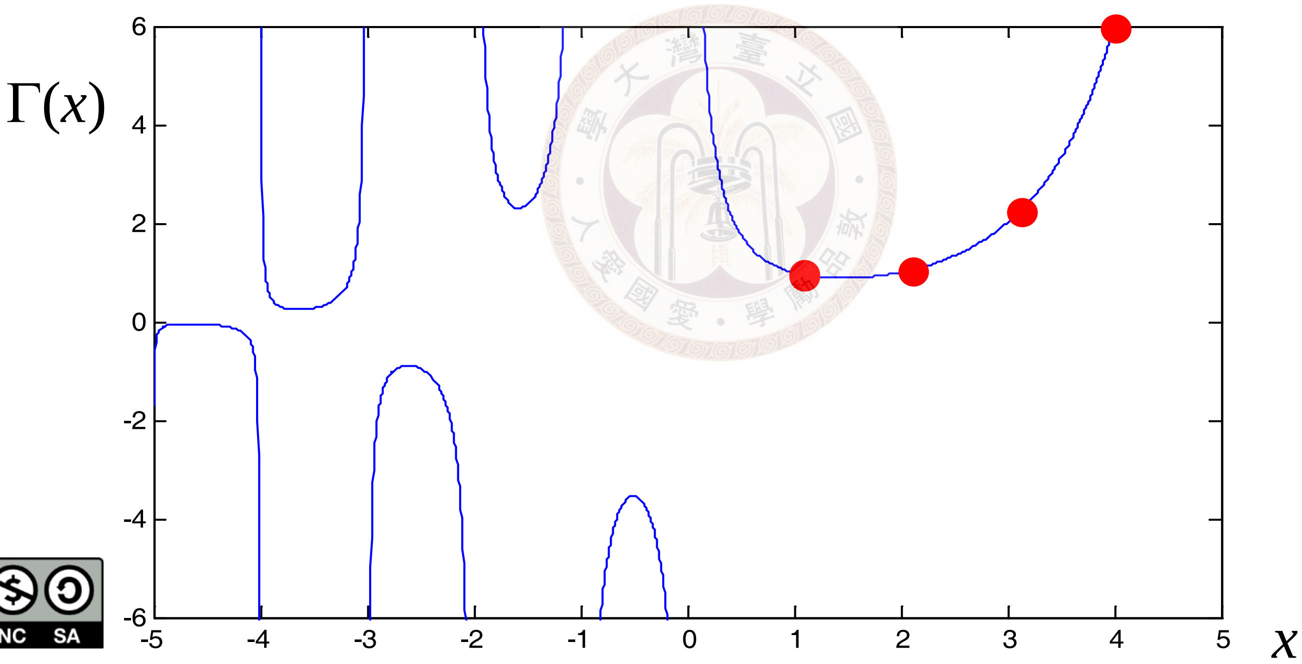
$\Gamma(1) = 0! = 1$

(2) $\Gamma(x+1) = x\Gamma(x)$

參照課本 Appendix 1

(3) $\Gamma(n) \to \infty$ when $n$ is a negative integer or $n = 0$

(4) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ $\qquad$ $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$



# 6.3.1.1 回到 Solving for Bessel function

$$c_{2n} = \frac{(-1)^n c_0}{2^{2n} n!(1+v)(2+v)(3+v)\cdots\cdots(n+v)}$$

when $r = v$

Set $c_0 = \frac{1}{2^v \Gamma(1+v)}$

$$c_{2n} = \frac{(-1)^n}{2^{2n+v} n!(1+v)(2+v)(3+v)\cdots\cdots(n+v)\Gamma(1+v)}$$

$$= \frac{(-1)^n}{2^{2n+v} n!(2+v)(3+v)\cdots\cdots(n+v)\Gamma(2+v)}$$

$$= \frac{(-1)^n}{2^{2n+v} n!(3+v)\cdots\cdots(n+v)\Gamma(3+v)}$$

$$\vdots$$

$$= \frac{(-1)^n}{2^{2n+v} n!\Gamma(n+v+1)}$$

$$\Gamma(2+v) = (1+v)\Gamma(1+v)$$

$$\Gamma(3+v) = (2+v)\Gamma(2+v)$$

同理，當 $r=-v$ set $c_0 = \dfrac{1}{2^{-v}\Gamma(1-v)}$

$$c_{2n} = \frac{(-1)^n}{2^{2n-v} n!\Gamma(n-v+1)}$$

Two independent solutions of the Bessel's equation

代入 $\sum_{n=0}^{\infty} c_n x^{n+r}$

When $r = v$ $$J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+v+n)} \left(\frac{x}{2}\right)^{2n+v}$$

When $r = -v$ $$J_{-v}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1-v+n)} \left(\frac{x}{2}\right)^{2n-v}$$

稱作 Bessel functions of the first kind of order $v$ and $-v$

## 6.3.1.3 Bessel function of the second kind

注意，兩個 roots 的差為 $2v$

(1) 當 $2v$ 不為整數時，Bessel's equation 的解即為

$$c_1 J_v(x) + c_2 J_{-v}(x) \quad (\text{也可表示成 } c_1 J_v(x) + c_2 Y_v(x))$$

(2) 當 $2v$ 為整數，但 $v = m + 1/2$ ($m$ 是一個整數) 時，Bessel's equation 的解亦為 $c_1 J_v(x) + c_2 J_{-v}(x)$ (也可表示成 $c_1 J_v(x) + c_2 Y_v(x)$)

(3) 當 $2v$ 為整數，且 $v$ 是一個整數時， Bessel's equation 的解為

$$c_1 J_v(x) + c_2 Y_v(x)$$

$Y_v(x)$: Bessel function of the second kind of order $v$

(見後頁)

$Y_v(x)$: Bessel function of the second kind of order $v$

$$Y_v(x) = \frac{\cos v\pi J_v(x) - J_{-v}(x)}{\sin v\pi}$$

當 $m$ 為整數時，$Y_m(x)$ 定義成

$$Y_m(x) = \lim_{v\to m} \frac{\cos v\pi J_v(x) - J_{-v}(x)}{\sin v\pi}$$

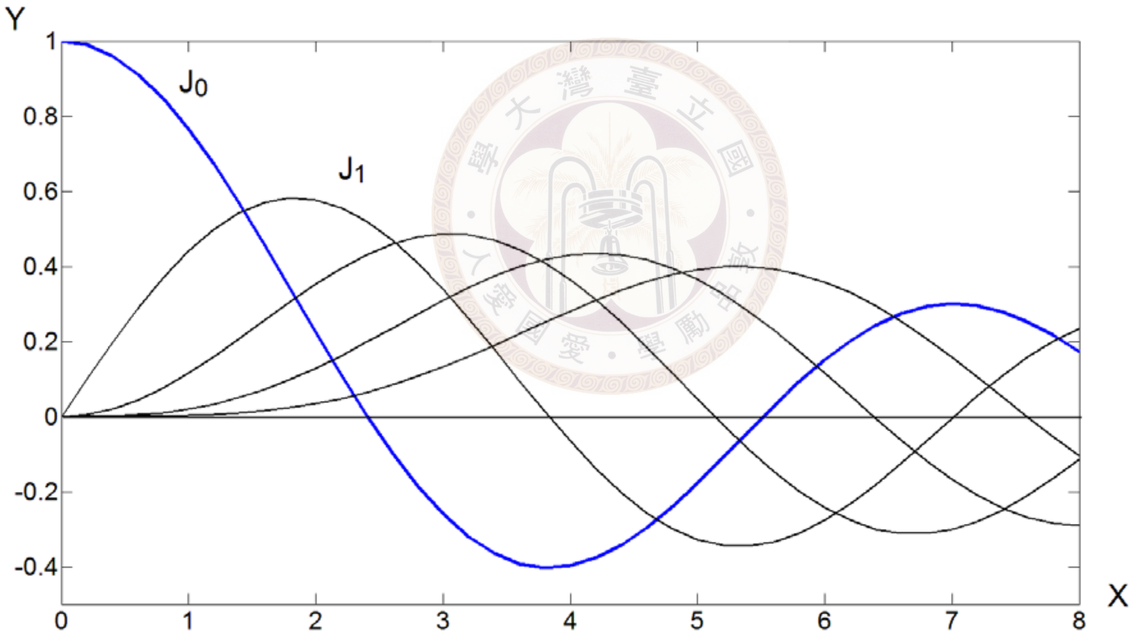
用 L'Hopital's rule 來算

$$Y_m(x) = \lim_{v\to m} \frac{-\pi \sin v\pi J_v(x) + \cos v\pi \frac{\partial}{\partial v} J_v(x) - \frac{\partial}{\partial v} J_{-v}(x)}{\pi \cos v\pi}$$

# 6.3.1.4 Bessel function of the 1st kind (order $m$ 為整數時)的性質

(1) $J_0(0) = 1, \qquad J_m(0) = 0$ for $m \neq 0$

(2) Zero crossing 的位置，隨著 $m$ 增加而越來越遠 (見 Table 6.3.1)



(3) $J_m(-x) = (-1)^m J_m(x)$ when $m$ is an integer

(4) $J_{-m}(x) = (-1)^m J_m(x)$ when $m$ is an integer

(5) $\frac{d}{dx}\left[x^{-v} J_v(x)\right] = -x^{-v} J_{v+1}(x)$

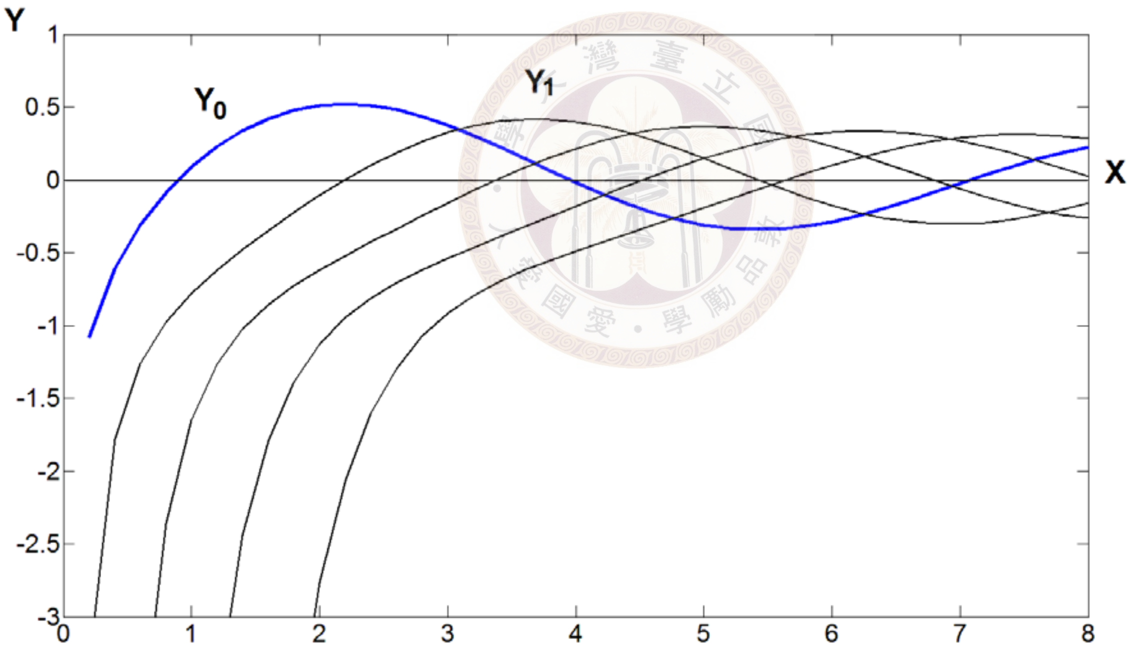
(6) $\frac{d}{dx}\left[x^{v} J_v(x)\right] = x^{v} J_{v-1}(x)$

見 Example 5, text page 246

# 6.3.1.5 Bessel function of the 2nd kind (order *m* 為整數時)的性質

(1) $\lim_{x \to 0} Y_m(x) = -\infty$

(2) Zero crossing 的位置，隨著 *m* 增加而越來越遠



## 6.3.1.6 Bessel's equation 的變型

$$x^2y'' + xy' + (x^2 - v^2)y = 0$$ 解：$c_1J_v(x) + c_2Y_v(x)$

(A) $$x^2y'' + xy' + (\alpha^2x^2 - v^2)y = 0$$ 解：$c_1J_v(\alpha x) + c_2Y_v(\alpha x)$

Proof: Set $t = \alpha x$

$$\frac{dy}{dx} = \frac{dt}{dx}\frac{dy}{dt} = \alpha\frac{dy}{dt}$$

Similarly, $$\frac{d^2y}{dx^2} = \frac{dt}{dx}\frac{d}{dt}\left(\frac{dy}{dx}\right) = \alpha\frac{d}{dt}\left(\alpha\frac{dy}{dt}\right) = \alpha^2\frac{d^2y}{dt^2}$$

原式 = $$x^2y'' + xy' + (\alpha^2x^2 - v^2)y = \frac{t^2}{\alpha^2}\alpha^2\frac{d^2y}{dt^2} + \frac{t}{\alpha}\alpha\frac{dy}{dt} + (\alpha^2\frac{t^2}{\alpha^2} - v^2)y$$

$$= \boxed{t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - v^2)y = 0}$$ → 對 $t$ 而言是 Bessel equation

$$y = c_1J_v(t) + c_2Y_v(t) = c_1J_v(\alpha x) + c_2Y_v(\alpha x)$$

(B) modified Bessel equation of order *v*

$$x^2y'' + xy' - (x^2 + v^2)y = 0$$

解：$c_1 I_v(x) + c_2 K_v(x)$

其中 $I_v(x) = i^{-v} J_v(ix)$ 稱作是 modified Bessel function of the first kind of order *v*

$$K_v(x) = \frac{\pi}{2}\frac{I_{-v}(x) - I_v(x)}{\sin v\pi}$$

稱作是 modified Bessel function of the second kind of order *v*

當 *v* 為整數時，也是取 limit

(C) $$x^2y'' + (1-2a)xy' + (b^2c^2x^{2c} + a^2 - p^2c^2)y = 0$$

解：$$y = x^a\left[c_1J_p(bx^c) + c_2Y_p(bx^c)\right]$$

式子有點複雜，但可以用來解許多物理上的問題

Example 3 (text page 244)

$$xy'' + 3y' + 9y = 0$$

$$\downarrow$$

$$x^2y'' + 3xy' + 9xy = 0$$

# 6.3.1.7 Spherical Bessel Functions

$J_v(x)$ 當 $v = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \cdots\cdots$ 時，稱作為 spherical Bessel functions

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

# 6.3.2 Legendre's Equation

## 6.3.2.1 Legendre's Equation

$$(1-x^2)y''-2xy'+n(n+1)y=0$$

$y(x)=\sum_{k=0}^{\infty} c_k x^k$ 代入，得出 (過程見課本 pages 248, 249)

二個 linearly independent 的解分別為

$$y_1(x)=c_0\left[1-\frac{n(n+1)}{2!}x^2+\frac{(n-2)n(n+1)(n+3)}{4!}x^4 -\frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!}x^6+\cdots\cdots\right]$$

$$y_2(x)=c_0\left[x-\frac{(n-1)(n+2)}{3!}x^3+\frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 -\frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!}x^7+\cdots\cdots\right]$$

(a) When $n$ is not an integer, both the two solutions have infinite number of terms.

(b) When $n$ is an even integer, $y_1(x)$ has finite number of terms.

In $y_1(x)$, the coefficient of $x^k$ is zero when $k > n$.

(c) When $n$ is an odd integer, $y_2(x)$ has finite number of terms.

In $y_2(x)$, the coefficient of $x^k$ is zero when $k > n$.

$y_1(x)$ when $n$ is an even integer and $y_2(x)$ when $n$ is an odd integer are called the Legendre polynomials (denoted by $P_n(x)$).

通常選

$$c_0 = (-1)^{n/2} \frac{1 \cdot 3 \cdots\cdots (n-1)}{2 \cdot 4 \cdots\cdots n} \qquad c_1 = (-1)^{(n-1)/2} \frac{1 \cdot 3 \cdots\cdots n}{2 \cdot 4 \cdots\cdots (n-1)}$$

(讓 $P_n(1)$ 一律等於 1)

由 $y_1(x)$

$$P_0(x) = 1$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x + 3)$$

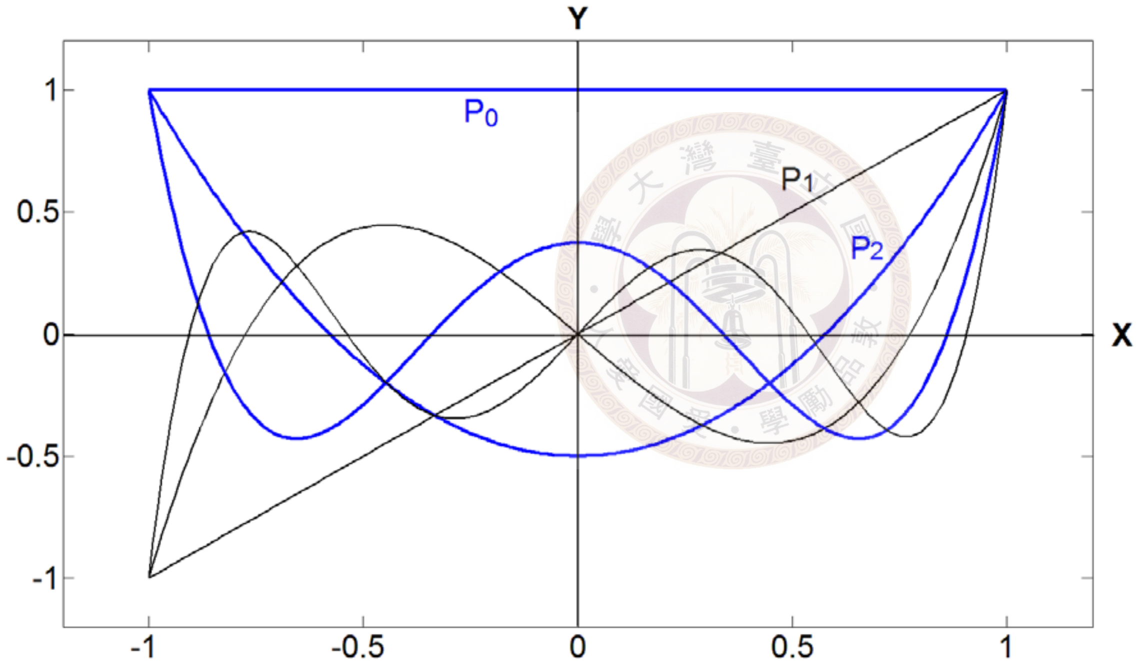
由 $y_2(x)$

$$P_1(x) = x$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

# Legendre polynomials




Interval:

$x \in [-1, 1]$

## 6.3.2.2 Properties of Legendre Polynomials

(1) $P_n(-x) = (-1)^n P_n(x)$ even / odd symmetry

(2) $P_n(1) = 1$ $P_n(-1) = (-1)^n$

(3) $P_n(0) = 0$ when $n$ is odd

(4) $P_n'(0) = 0$ when $n$ is even

(5) $(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$ recursive relation

(6) $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}\left(x^2-1\right)^n$ Rodrigues' formula

(7) $\int_{-1}^{1} P_m(x) P_n(x) dx = 0$ If $m \neq n$ orthogonality property

(8) 若任何在 $x \in [-1, 1]$ 區間為 continuous 的函式 $f(x)$

皆可表示為

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

由於 $\int f(x) P_m(x) dx = \sum_{n=0}^{\infty} a_n \int P_n(x) P_m(x) dx = a_n \int P_m(x) P_m(x) d$

根據 orthogonality property

所以 $a_n = \dfrac{\int f(x) P_m(x) dx}{\int P_m(x) P_m(x) d}$

Orthogonality property 才是 Legendre polynomials 最重要的性質

# 6.3.2.3 補充：其他常見的 orthogonal polynomial

- Chebychev polynomials　電子學和 filter design 常用

Solutions of $(1-x^2)P''_m(x) - xP'_m(x) + n^2 P_m(x) = 0$

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} P_m(x) P_n(x)\, dx = 0$$

$$P_n(\cos\theta) = C_n \cos n\theta$$

- Hermite polynomials　電磁波、光學、頻譜分析常用

Solutions of $P''_m(x) - xP'_m(x) + n P_m(x) = 0$

$$\int_{-\infty}^{\infty} e^{-x^2} P_m(x) P_n(x)\, dx = 0$$

## 6.3.3 Section 6-3 需要注意的地方：

(1) 概念簡單，但是定義，性質，和數學式甚多

擇要而學即可

(2) 要了解 Gamma function

# Review of Chapter 6

解法適用範圍： Linear DE，且 coefficients 最好為 polynomials

$$y^{(n)}+P_{n-1}(x)y^{(n-1)}+\cdots\cdots+P_1(x)y'+P_0(x)y=0$$

- 當 $P_m(x)$ 在 $x = x_0$ 時為 analytic

  $x_0$ 為 ordinary point $\quad y(x)=\sum_{n=0}^{\infty}c_n(x-x_0)^n$ 代入

- 當 $P_m(x)$ 在 $x = x_0$ 時不為 analytic

  但是 $(x-x_0)^{n-m}P_m(x)$ 在 $x = x_0$ 時為 analytic

  $x_0$ 為 regular singular point $\quad y(x)=\sum_{n=0}^{\infty}c_n(x-x_0)^{n+r}$ 代入

  有時，另一個解為 $\quad Cy_1(x)\ln x+\sum_{n=0}^{\infty}b_n(x-x_0)^{n+r_2}$

# Exercise for practice

| | |
|---|---|
| Sec. 6-1: | 2, 3, 10, 14, 16, 23, 28, 30, 33, 36 |
| Sec. 6-2: | 4, 9, 13, 22, 28, 29, 31, 33, 36 |
| Sec. 6-3: | 3, 9, 25, 27, 29, 37, 46, 47 |
| Review 6 | 7, 10, 14, 19, 20, 22 |

# 版權聲明

| 頁碼 | 作品 | 版權標示 | 作者/來源 |
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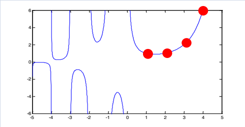
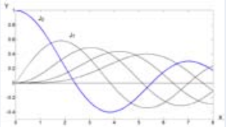
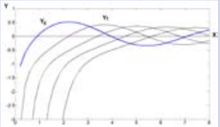
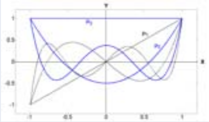
| 387 |  | CC BY NC SA | 國立臺灣大學 電信工程所 丁建均 教授 |
| 392 |  | CC BY NC SA | 國立臺灣大學 電信工程所 丁建均 教授 |
| 394 |  | CC BY NC SA | 國立臺灣大學 電信工程所 丁建均 教授 |
| 402 |  | CC BY NC SA | 國立臺灣大學 電信工程所 丁建均 教授 |