4-4 Undetermined Coefficients – Superposition Approach

This section introduces some method of "guessing" the particular solution.

4-4-1 方法適用條件

(1) (2)

Suitable for linear and constant coefficient DE.

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y = g(x)$$

(3) $g(x)$, $g'(x)$, $g''(x)$, $g'''(x)$, $g^{(4)}(x)$, $g^{(5)}(x)$, ...contain finite number of terms.

4-4-2 方法

把握一個原則：

$g(x)$ 長什麼樣子，particular solution 就應該是什麼樣子.

記熟下一頁的規則

(計算時要把 A, B, C, … 這些 unknowns 解出來)

Trial Particular Solutions (from text page 144)

$g(x)$	Form of y_p
1 (any constant)	A
$5x + 7$	$Ax + B$
$3x^2 - 2$	$Ax^2 + Bx + C$
$x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
$\sin 4x$	$A \cos 4x + B \sin 4x$
$\cos 4x$	$A \cos 4x + B \sin 4x$
e^{5x}	$A e^{5x}$
$(9x-2) e^{5x}$	$(Ax + B) e^{5x}$
$x^2 e^5$	$(Ax^2 + Bx + C) e^{5x}$
$e^3 \sin 4x$	$A e^3 \cos 4x + B e^3 \sin 4x$
$5x^2 \sin 4x$	$(Ax^2 + Bx + C) \cos 4x + (Ex^2 + Fx + G) \sin 4x$
$xe^3 \cos 4x$	$(Ax + B) e^3 \cos 4x + (Cx + E) e^3 \sin 4x$

It comes from the "form rule". See page 196.

$$g(x) = e^{2x} + xe^{3x} \qquad y_p = ?$$

$$g(x) = \cos(x) + x^2 \sin(2x) \qquad y_p = ?$$

$$g(x) = \cosh(2x) \qquad y_p = ?$$

4-4-3 Examples

Example 2 $y'' - y' + y = 2\sin 3x$ (text page 142)

Step 1: find the solution of the associated homogeneous equation

Guess

Step 2: particular solution $y_p = A\cos 3x + B\sin 3x$

$$y_p' = -3A\sin 3x + 3B\cos 3x$$

$$y_p'' = -9A\cos 3x - 9B\sin 3x$$

$$y_p'' - y_p' + y_p = (-8A - 3B)\cos 3x + (3A - 8B)\sin 3x = 2\sin 3x$$

$$\begin{cases} -8A - 3B = 0 \\ 3A - 8B = 2 \end{cases} \Longrightarrow A = 6/73,\ B = -16/73$$

$$y_p = \frac{6}{73}\cos 3x - \frac{16}{73}\sin 3x$$

Step 3: General solution:

$$y = e^{x/2}\left(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right) + \frac{6}{73}\cos 3x - \frac{16}{73}\sin 3x$$

Example 3 $y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$ (text page 143)

Step 1: Find the solution of

$$y'' - 2y' - 3y = 0.$$

$$y_c = c_1 e^{3x} + c_2 e^{-x}$$

Step 2: Particular solution

$$y'' - 2y' - 3y = 4x - 5$$

guess

$$y_{p_1} = Ax + B$$

$$y'_{p_1} = A$$

$$y''_{p_1} = 0$$

$$-3Ax - 2A - 3B = 4x - 5$$

$$A = -\frac{4}{3}, \quad B = \frac{23}{9}$$

$$y_{p_1} = -\frac{4}{3}x + \frac{23}{9}$$

$$y'' - 2y' - 3y = 6xe^{2x}$$

guess

$$y_{p_2} = Cxe^{2x} + Ee^{2x}$$

$$y'_{p_2} = 2Cxe^{2x} + Ce^{2x} + 2Ee^{2x}$$

$$y''_{p_2} = 4Cxe^{2x} + 4Ce^{2x} + 4Ee^{2x}$$

$$-3Cxe^{2x} + (2C - 3E)e^{2x} = 6xe^{2x}$$

$$C = -2, \quad E = -\frac{4}{3}$$

$$y_{p_2} = -(2x + \frac{4}{3})e^{2x}$$

Particular solution

$$y_p = y_{p_1} + y_{p_2} = -\frac{4}{3}x + \frac{23}{9} - (2x + \frac{4}{3})e^x$$

Step 3: General solution

$$y = y_c + y_p$$

$$\boxed{y = c_1 e^{3x} + c_2 e^{-x} - \frac{4}{3}x + \frac{23}{9} - (2x + \frac{4}{3})e^{2x}}$$

4-4-4 方法的解釋

Form Rule: y_p should be a linear combination of $g(x)$, $g'(x)$, $g''(x)$, $g'''(x)$, $g^{(4)}(x)$, $g^{(5)}(x)$, ………………

Why? 如此一來，在比較係數時才不會出現多餘的項

When $g(x) = x^n$

$$x^n \to x^{n-1} \to x^{n-2} \to x^{n-3} \to \cdots\cdots \to 1 \to 0$$

$$y_p = A_n x^n + A_{n-1} x^{n-1} + A_{n-2} x^{n-2} + \ldots\ldots + A_0$$

When $g(x) = \cos kx$

$$\cos kx \to \sin kx$$

$$y_p = A_1 \cos kx + A_2 \sin kx$$

When $g(x) = \exp(kx)$

$$e^{kx}$$

$$y_p = A \exp(kx)$$

When $g(x) = x^n \exp(kx)$

$$g'(x) = nx^{n-1}e^{kx} + kx^n e^{kx}$$

$$g''(x) = n(n-1)x^{n-2}e^{kx} + 2nkx^{n-1}e^{kx} + k^2x^n e^{kx}$$

$$g'''(x) = n(n-1)(n-2)x^{n-3}e^{kx} + 3kn(n-1)x^{n-2}e^{kx}$$

$$+3k^2nx^{n-1}e^{kx} + k^3x^n e^{kx}$$

$$\vdots$$

$$\vdots$$

會發現 $g(x)$ 不管多少次微分，永遠只出現

$$x^n e^{kx},\ x^{n-1}e^{kx},\ x^{n-2}e^{kx},\ x^{n-3}e^{kx},\ \cdots\cdots,\ e^{kx}$$

$$\boxed{y_p = c_n x^n e^{kx} + c_{n-1}x^{n-1}e^{kx} + c_{n-2}x^{n-2}e^{kx} + \cdots\cdots + c_0 e^{kx}}$$

4-4-5 Glitch of the method:

Example 4 $y'' - 5y' + 4y = 8e^x$ (text page 143)

Particular solution guessed by Form Rule:

$$y_p = Ae^x$$

$$y_p'' - 5y_p' + 4y_p = Ae^x - 5Ae^x + 4Ae^x = 8e^x$$

$$0 = 8e^x$$ (no solution)

Why?

Glitch condition 1: The **particular solution** we guess belongs to the complementary function.

For Example 4 $y'' - 5y' + 4y = 8e^x$

Complementary function $y_c = c_1 e^x + c_2 e^{4x}$ $Ae^x \in y_c$

解決方法：再乘一個 x

$$y_p = Axe^x$$

$$y_p' = Axe^x + Ae^x$$

$$y_p'' = Axe^x + 2Ae^x$$

$$y_p'' - 5y_p' + 4y_p = -3Ae^x = 8e^x \Longrightarrow A = -8/3$$

$$y_p = -\frac{8}{3}xe^x$$

$$y = c_1 e^x + c_2 e^{4x} - \frac{8}{3}xe^x$$

Example 7

$$y'' - 2y' + y = e^x$$

(text page 145)

$$y_c = c_1 e^x + c_2 x e^x$$

From Form Rule, the particular solution is Ae^x

$$Ae^x \in y_c$$

$$Axe^x \in y_c$$

如果乘一個 x 不夠，則再乘一個 x

$$y_p = Ax^2 e^x$$

$$y_p' = (Ax^2 + 2Ax)e^x$$

$$y_p'' = (Ax^2 + 4Ax + 2A)e^x$$

$$y_p'' - 2y_p' + y_p = 2Ae^x = e^x \quad \Longrightarrow \quad A = 1/2$$

$$y_p = x^2 e^x / 2$$

$$y = c_1 e^x + c_2 x e^x + x^2 e^x / 2$$

Example 8 (text page 146)

$$y'' + y = 4x + 10\sin x \qquad y(\pi) = 0 \qquad y'(\pi) = 2$$

Step 1 $\quad y_c = c_1 \cos x + c_2 \sin x$

Step 2 $\quad y_p = Ax + B + Cx\sin x + Ex\cos x$

$$y_p = 4x - 5x\cos x$$

注意： sinx, cosx 都要乘上 x

Step 3 $\quad y = c_1 \cos x + c_2 \sin x + 4x - 5x\cos x$

Step 4 $\quad$ Solving c_1 and c_2 by initial conditions (最後才解 IVP)

$$y(\pi) = -c_1 + 4\pi + 5\pi = 0 \Longrightarrow c_1 = 9\pi$$

$$y' = -c_1 \sin x + c_2 \cos x + 4 - 5\cos x + 5x\sin x$$

$$y'(\pi) = -c_2 + 9 = 2 \Longrightarrow c_2 = 7$$

$$y = 9\pi\cos x + 7\sin x + 4x - 5x\cos x$$

Example 11 (text page 147)

$$y^{(4)} + y''' = 1 - x^2 e^{-x}$$

$$y_c = \boxed{c_1} + c_2 x + c_3 x^2 + \boxed{c_4 e^{-x}}$$

From Form Rule

$$y_p = \boxed{A} + Bx^2 e^{-x} + Cxe^{-x} + \boxed{Ee^{-x}}$$

y_p 只要有一部分和 y_c 相同就作修正

修正

$$y_p = Ax^3 + Bx^3 e^{-x} + Cx^2 e^{-x} + Exe^{-x}$$

乘上 x^3

乘上 x

If we choose $y_p = A + Bx^2 e^{-x} + Cxe^{-x} + Ee^{-x}$

$$y_p^{(4)} + y_{(p)}''' = -2Bxe^{-x} + (6B - C)e^{-x} = 1 - x^2 e^{-x}$$

沒有 1, x^2e^{-x} 兩項，不能比較係數，無解

If we choose $y_p = Ax^3 + Bx^2e^{-x} + Cxe^{-x} + Ee^{-x}$

$$y_p^{(4)} + y'''_{(p)} = 6A - 2Bxe^{-x} + (6B - C)e^{-x} = 1 - x^2e^{-x}$$

沒有 x^2e^{-x} 這一項，不能比較係數，無解

If we choose $y_p = Ax^3 + Bx^3e^{-x} + Cx^2e^{-x} + Exe^{-x}$

$$y_p^{(4)} + y'''_{(p)}$$

$$= 6A - 3Bx^2e^{-x} + (18B - 2C)xe^{-x} + (-18B + 6C - E)e^{-x}$$

$$= 1 - x^2e^{-x}$$

$A = 1/6$, $B = 1/3$, $C = 3$, $E = 12$

$$y_p = \frac{1}{6}x^3 + \frac{1}{3}x^3e^{-x} + 3x^2e^{-x} + 12xe^{-x}$$

$$y = c_1 + c_2x + c_3x^2 + c_4e^{-x} + \frac{1}{6}x^3 + \frac{1}{3}x^3e^{-x} + 3x^2e^{-x} + 12xe^{-x}$$

Glitch condition 2: $g(x)$, $g'(x)$, $g''(x)$, $g'''(x)$, $g^{(4)}(x)$, $g^{(5)}(x)$, …………… contain **infinite number of terms**.

If $g(x) = \ln x$

$$\ln x \to \frac{1}{x} \to \frac{1}{x^2} \to \frac{1}{x^3} \to \cdots\cdots$$

If $g(x) = \exp(x^2)$

$$g'(x) \to 2xe^{x^2}$$
$$g''(x) \to (4x^2+2)e^{x^2}$$
$$g'''(x) \to (8x^3+12x)e^{x^2}$$

$$\vdots$$

$$\vdots$$

4-4-6 本節需要注意的地方

(1) 記住 Table 4.1 的 particular solution 的假設方法
(其實和 "form rule" 有相密切的關聯)

(2) 注意 "glitch condition"

另外，"同一類" 的 term 要乘上相同的東西 (參考 Example 11)

(3) 所以要先算 complementary function，再算 particular solution

(4) 同樣的方法，也可以用在 1st order 的情形

(5) 本方法只適用於 linear, constant coefficient DE

4-5 Undetermined Coefficients – Annihilator Approach

For a linear DE:

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y = g(x)$$

Annihilator Operator:

能夠「殲滅」$g(x)$ 的 operator

4-5-1 方法適用條件

(1) Linear， (2) Constant coefficients

(3) $g(x)$, $g'(x)$, $g''(x)$, $g'''(x)$, $g^{(4)}(x)$, $g^{(5)}(x)$, ………contain finite number of terms.

4-5-2 Find the Annihilator

Example 1: (text page 151)

$g(x) = 1 - 5x^2 + 8x^3$ ⟶ annihilator: D^4 $\quad D^k g(x) = \frac{d^k}{dx^k} g(x)$

$g(x) = e^{-3x}$ ⟶ annihilator: $D + 3$

$$\frac{d}{dx} g(x) + 3g(x) = 0$$

$$g(x) = 4e^{2x} - 10xe^{2x} \longrightarrow \text{annihilator: } (D-2)^2$$

$$(D-2)^2 = D^2 - 4D + 4$$

$$\frac{d^2}{dx^2} g(x) - 4\frac{d}{dx} g(x) + 4g(x) = 0$$

註：當 coefficient 為 constants 時，function of D 的計算方式和 function of x 的計算方式相同

$$(x-2)^2 = x^2 - 4x + 4$$

$$\Rightarrow (D-2)^2 = D^2 - 4D + 4$$

General rule 1:

If $g(x)=\left(a_n x^n + a_{n-1}x^{n-1} + \cdots\cdots + a_0\right)e^{\alpha x}$

then the annihilator is $[D-\alpha]^{n+1}$

注意： annihilator 和 $a_0, a_1, \ldots\ldots, a_n$ 無關

只和 α, n 有關

General rule 2:

If $g(x) = \left(a_n x^n + a_{n-1}x^{n-1} + \cdots\cdots + a_0\right) e^{\alpha x} \left(b_1 \cos \beta x + b_2 \sin \beta x\right)$

$b_1 \neq 0$ or $b_2 \neq 0$

then the annihilator is $\left[D^2 - 2\alpha D + \left(\alpha^2 + \beta^2\right)\right]^{n+1}$

Example 2: (text page 151) $g(x) = 5e^{-x}\cos 2x - 9e^{-x}\sin 2x$

annihilator $D^2 + 2D + 5$

Example 5: (text page 154) $g(x) = x\cos x - \cos x$

annihilator $\left[D^2 + 1\right]^2$

Example 6: (text page 155) $g(x) = 10e^{-2x}\cos x$

annihilator $D^2 + 4D + 5$

General rule 3:

If $g(x) = g_1(x) + g_2(x) + \ldots\ldots + g_k(x)$

$L_h[g_h(x)] = 0$ but $L_h[g_m(x)] \neq 0$ if $m \neq h$,

then the annihilator of $g(x)$ is the product of L_h $(h = 1 \sim k)$

$$L_k L_{k-1} \cdots L_2 L_1$$

Proof:

$$L_k L_{k-1} \cdots L_3 L_2 L_1 [g_1 + g_2 + g_3 + \cdots\cdots + g_k]$$
$$= L_k L_{k-1} \cdots L_3 L_2 L_1 g_1 + L_k L_{k-1} \cdots L_3 L_2 L_1 g_2 +$$
$$L_k L_{k-1} \cdots L_3 L_2 L_1 g_3 + \cdots\cdots + L_k L_{k-1} \cdots L_3 L_2 L_1 g_k$$

$$L_k L_{k-1} \cdots L_3 L_2 L_1 g_1 = L_k L_{k-1} \cdots L_3 L_2 [L_1 g_1] = 0$$
$$L_k L_{k-1} \cdots L_3 L_2 L_1 g_2 = L_k L_{k-1} \cdots L_3 L_1 [L_2 g_2] = 0$$

(因為 L_1, L_2 為 linear DE with constant coefficient, $L_1 L_2 = L_2 L_1$)

Similarly,

$$L_k L_{k-1} \cdots L_4 L_3 L_2 L_1 g_3 = L_k L_{k-1} \cdots L_4 L_2 L_1 [L_3 g_3] = 0$$

$$\vdots$$

$$\vdots$$

$$L_k L_{k-1} \cdots L_4 L_3 L_2 L_1 g_3 = L_{k-1} \cdots L_4 L_3 L_2 L_1 [L_k g_k] = 0$$

Therefore,

$$L_k L_{k-1} \cdots L_3 L_2 L_1 [g_1 + g_2 + g_3 + \cdots\cdots + g_k]$$

$$= 0 + 0 + 0 + \cdots\cdots + 0$$

$$= 0$$

Example 7 (text page 155)

$$g(x) = 5x^2 - 6x + 4x^2e^{2x} + 3e^{5x}$$

annihilator: D^3

annihilator: $(D-2)^3$

annihilator: $D-5$

annihilator of $g(x)$: $D^3(D-2)^3(D-5)$

4-5-3 Using the Annihilator to Find the Particular Solution

Step 2-1 Find the annihilator L_1 of $g(x)$

Step 2-2 如果原來的 linear & constant coefficient DE 是

$$L(y) = g(x)$$

那麼將 DE 變成如下的型態：

$$\boxed{L_1[L(y)] = L_1[g(x)] = 0}$$

(homogeneous linear & constant coefficient DE)

註： If $a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y = g(x)$

then $L = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0$

Step 2-3 Use the method in Section 4-3 to find the solution of

$$\boxed{L_1[L(y)] = 0}$$

Step 2-4 Find the particular solution.

The particular solution y_p is a solution of

$$L_1[L(y)] = 0$$

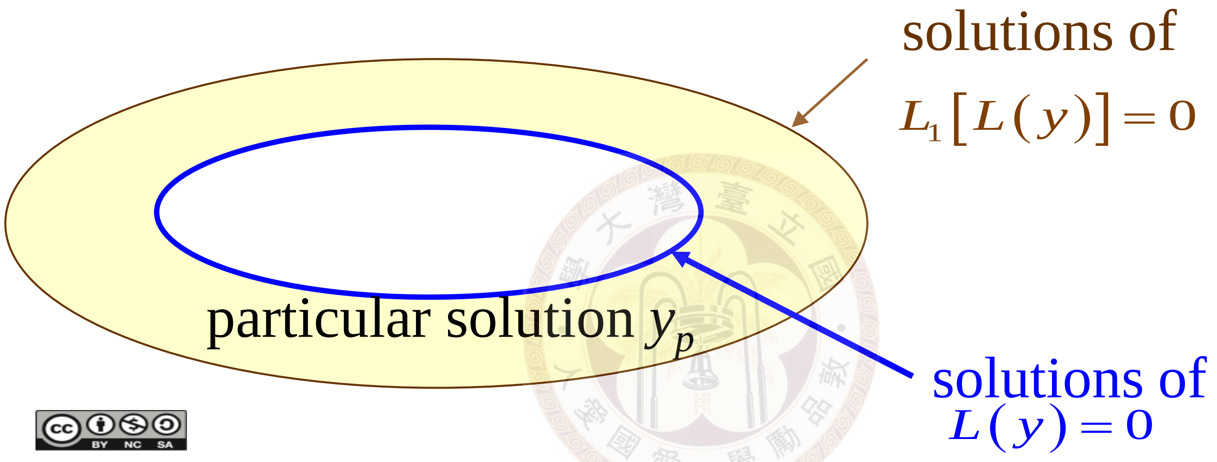
$$L(y) = 0$$

but not a solution of

(Proof): Since $L(y_p) = g(x)$, if $g(x) \neq 0$, $L(y_p)$ should be nonzero.

Moreover, $L_1\left[L(y_p)\right] = L_1[g(x)] = 0$.

Step 2-5 Solve the unknowns



particular solution y_p $\in$ solutions of $L_1[L(y)]=0$

$\notin$ solutions of $L(y)=0$

4-5-4 Examples

Example 3 (text page 153)

$$y'' + 3y' + 2y = 4x^2$$

Step 1: Complementary function

(solution of the associated homogeneous function)

$$y_c = c_1 e^{-x} + c_2 e^{-2x}$$

Step 2-1: Annihilation: D^3

$$L_1[L(y)] = L_1[g(x)] = 0$$

Step 2-2: $D^3(D^2 + 3D + 2)y = 0$

Step 2-3: auxiliary function $m^3(m^2 + 3m + 2) = 0$

roots: $m_1 = m_2 = m_3 = 0,\ m_4 = -1,\ m_5 = -2$

Solution for $L_1[L(\tilde{y})] = 0$:

$$\tilde{y} = \boxed{d_1 + d_2 x + d_3 x^2} + d_4 e^{-x} + d_5 e^{-2x}$$

移除和 complementary function 相同的部分

Step 2-4: particular solution $y_p = A + Bx + Cx^2$

$$y_p' = B + 2Cx$$
$$y_p'' = 2C$$

Step 2-5: $y_p'' + 3y_p' + 2y = 2Cx^2 + (2B + 6C)x + (2A + 3B + 2C) = 4x^2$

$$\begin{cases} 2C = 4 \\ 2B + 6C = 0 \\ 2A + 3B + 2C = 0 \end{cases} \Rightarrow \begin{matrix} C = 2 \\ B = -6 \\ A = 7 \end{matrix}$$

$$\boxed{y_p = 7 - 6x + 2x^2}$$

Step 3: $\boxed{y = y_c + y_p = c_1 e^{-x} + c_2 e^{-2x} + 7 - 6x + 2x^2}$

Example 4 (text page 154)

$$y'' - 3y' = 8e^{3x} + 4\sin x$$

Step 1: Complementary function

From auxiliary function, $m^2 - 3m = 0$, roots: 0, 3

$$y_c = c_1 + c_2 e^{3x}$$

Step 2-1: Find the annihilator

$D - 3$ annihilate $8e^{3x}$ but cannot annihilate $4\sin x$

$(D^2 + 1)$ annihilate $4\sin x$ but cannot annihilate $8e^{3x}$

$\Downarrow$

$(D - 3)(D^2 + 1)$ is the annihilator of $8e^{3x} + 4\sin x$

Step 2-2: $(D-3)(D^2+1)\left(D^2 - 3D\right)y = 0$

Step 2-3: auxiliary function: $(m-3)(m^2+1)\left(m^2-3m\right)$

$$= m(m-3)^2(m^2+1) = 0$$

易犯錯的地方

solution of $(D-3)(D^2+1)\left(D^2-3D\right)\tilde{y} = 0$:

$$\tilde{y} = d_1 + d_2 e^{3x} + \boxed{d_3 x e^{3x} + d_4 \cos x + d_5 \sin x}$$

Step 2-4: particular solution

$$y_p = d_3 x e^{3x} + d_4 \cos x + d_5 \sin x$$

⇓ 代回原式並比較係數

Step 2-5: $y_p = \frac{8}{3} x e^{3x} + \frac{6}{5}\cos x - \frac{2}{5}\sin x$

Step 3: general solution $\boxed{y = c_1 + c_2 e^{3x} + \frac{8}{3} x e^{3x} + \frac{6}{5}\cos x - \frac{2}{5}\sin x}$

4-5-5 本節要注意的地方

(1) 所以要先算 complementary function，再算 particular solution

(2) 若有兩個以上的 annihilator，選其中較簡單的即可

(3) 計算 auxiliary function 時有時容易犯錯

(4) $L_1[L(\tilde{y})]=0$ 的解和 $L(y)=0$ 的解不一樣。

(5) 這方法，只適用於 constant coefficient linear DE

(因為，還需借助 auxiliary function)

The thing that can be done by the annihilator approach can always be done by the "guessing" method in Section 4-4, too.

4-6 Variation of Parameters

4-6-1 方法的限制

The method can solve the particular solution for **any** linear DE

(1) May not have constant coefficients
(2) *g*(*x*) may not be of the special forms

$$a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_1(x)y'(x)+a_0(x)y=g(x)$$

4-6-2 Case of the 2nd order linear DE

$$a_2(x)y''(x)+a_1(x)y'(x)+a_0(x)y=g(x)$$

associated homogeneous equation: $a_n(x)y''(x)+a_1(x)y'(x)+a_0(x)y=0$

Suppose that the solution of the associated homogeneous equation is

$$c_1y_1(x)+c_2y_2(x)$$

Then the particular solution is assumed as:

$$y_p=u_1(x)y_1(x)+u_2(x)y_2(x)$$

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$$

代入原式後，總是可以簡化

$$y_p' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$$

$$y_p'' = u_1''y_1 + 2u_1'y_1' + u_1y_1'' + u_2''y_2 + 2u_2'y_2' + u_2y_2''$$

代入 $y''(x) + P(x)y'(x) + Q(x)y = f(x)$

$$P(x) = \frac{a_1(x)}{a_2(x)}, \quad Q(x) = \frac{a_0(x)}{a_2(x)}, \quad f(x) = \frac{g(x)}{a_2(x)}$$

zero zero

$$y_p'' + P(x)y_p' + Q(x)y_p = u_1[y_1'' + Py_1' + Qy_1] + u_2[y_2'' + Py_2' + Qy_2] + y_1u_1''$$
$$+ 2u_1'y_1' + y_2u_2'' + 2u_2'y_2' + P[y_1u_1' + y_2u_2']$$

$$y_p'' + P(x)y_p' + Q(x)y_p = f(x), \qquad y_p = u_1y_1 + u_2y_2$$

↓ 簡化

$$\frac{d}{dx}[y_1u_1' + y_2u_2'] + P[y_1u_1' + y_2u_2'] + y_1'u_1' + y_2'u_2' = f(x)$$

↓ 進一步簡化：

假設 $y_1u_1' + y_2u_2' = 0$

$$y_1'u_1' + y_2'u_2' = f(x)$$

聯立方程式

$$\begin{cases} y_1u_1' + y_2u_2' = 0 \\ y_1'u_1' + y_2'u_2' = f(x) \end{cases}$$

$$\begin{cases} y_1 u_1' + y_2 u_2' = 0 \\ y_1' u_1' + y_2' u_2' = f(x) \end{cases} \Longrightarrow \quad u_1' = \frac{W_1}{W} = -\frac{y_2 f(x)}{W}, \quad u_2' = \frac{W_2}{W} = \frac{y_1 f(x)}{W} \quad \Longrightarrow \quad u_1 = \int u_1'(x)\,dx, \quad u_2 = \int u_2'(x)\,dx$$

where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ $\quad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$ $\quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$

| |: determinant

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

可以和 1st order case (page 56) 相比較

4-6-3 Process for the 2nd Order Case

Step 2-1 變成 standard form

$$y''(x)+P(x)y'(x)+Q(x)y=f(x)$$

Step 2-2

$$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \qquad W_1=\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} \qquad W_2=\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

Step 2-3

$$u_1'=\frac{W_1}{W} \qquad u_2'=\frac{W_2}{W}$$

Step 2-4

$$u_1=\int u_1'(x)\,dx \qquad u_2=\int u_2'(x)\,dx$$

Step 2-5

$$y_p(x)=u_1(x)y_1(x)+u_2(x)y_2(x)$$

4-6-4 Examples

Example 1 (text page 159)

$$y'' - 4y' + 4y = (x+1)e^{2x}$$

Step 1: solution of $y'' - 4y' + 4y = 0$:

$$y_c = c_1 e^{2x} + c_2 x e^{2x}$$

Step 2-2: $y_p = u_1 y_1 + u_2 y_2$, $y_1 = e^{2x}$, $y_2 = xe^{2x}$

$$W = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = e^{4x} \qquad W_1 = \begin{vmatrix} 0 & xe^{2x} \\ (x+1)e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = -(x+1)xe^{4x}$$

$$W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (x+1)e^{2x} \end{vmatrix} = (x+1)e^{4x}$$

Step 2-3: $u_1' = \dfrac{W_1}{W} = -x^2 - x$ $\qquad u_2' = \dfrac{W_2}{W} = x + 1$

Step 2-4: $u_1 = \int u_1' dx = \int (-x^2 - x) dx = -\frac{1}{3}x^3 - \frac{1}{2}x^2 + \cancel{c_1}$

$u_2 = \int u_2' dx = \int (x+1) dx = \frac{1}{2}x^2 + x + \cancel{c_2}$

Step 2-5: $y_p = (-\frac{1}{3}x^3 - \frac{1}{2}x^2)e^{2x} + (\frac{1}{2}x^2 + x)xe^{2x} = (\frac{1}{6}x^3 + \frac{1}{2}x^2)e^{2x}$

Step 3: $\boxed{y = c_1 e^{2x} + c_2 x e^{2x} + (\frac{1}{6}x^3 + \frac{1}{2}x^2)e^{2x}}$

Example 2 (text page 159) $4y'' + 36y = \csc 3x$

$$f(x) = \csc 3x / 4$$

Note: (a) $\frac{1}{12}\int \frac{\cos 3x}{\sin 3x} dx$ 的算法

(b) Interval (0, π/6) 改為(0, π/3)

Example 3 (text page 160) $y'' - y = 1/x$

$$f(x) = 1/x$$

Note: $\int \frac{e^x}{x} dx$ 沒有 analytic 的解

所以直接表示成 $\int_{x_0}^{x} \frac{e^t}{t} dt$ (複習 page 43)

4-6-5 Case of the Higher Order Linear DE

$$a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_1(x)y'(x)+a_0(x)y=g(x)$$

Solution of the associated homogeneous equation:

$$y_c=c_1y_1(x)+c_2y_2(x)+c_3y_3(x)+\cdots\cdots+c_ny_n(x)$$

The particular solution is assumed as:

$$y_p=u_1(x)y_1(x)+u_2(x)y_2(x)+u_3(x)y_3(x)+\cdots\cdots+u_n(x)y_n(x)$$

$$u_k'(x)=\frac{W_k}{W} \Longrightarrow u_k(x)=\int u_k'(x)d$$

$$u_k'(x) = \frac{W_k}{W}$$

$$W = \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ y_1' & y_2' & y_3' & \cdots & y_n' \\ y_1'' & y_2'' & y_3'' & \cdots & y_n'' \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & y_3^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

$$W_k = \begin{vmatrix} y_1 & y_2 & \cdots & y_{k-1} & 0 & y_{k+1} & \cdots & y_n \\ y_1' & y_2' & \cdots & y_{k-1}' & 0 & y_{k+1}' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_{k-1}^{(n-2)} & 0 & y_{k+1}^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_{k-1}^{(n-1)} & f(x) & y_{k+1}^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

$$f(x) = g(x)/a_n(x)$$

W_k: replace the k^{th} column of W by $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(x) \end{bmatrix}$

$$f(x) = \frac{g(x)}{a_n(x)}$$

For example, when $n = 3$,

$$W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ f(x) & y_2'' & y_3'' \end{vmatrix} \qquad W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & f(x) & y_3'' \end{vmatrix} \qquad W_3 = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & f(x) \end{vmatrix}$$

4-6-6 Process of the Higher Order Case

Step 2-1 變成 standard form

$$y^{(n)}(x)+\frac{a_{n-1}(x)}{a_n(x)}y^{(n-1)}(x)+\cdots+\frac{a_1(x)}{a_n(x)}y'(x)+\frac{a_0(x)}{a_n(x)}y=\frac{g(x)}{a_n(x)}$$

Step 2-2 Calculate W, W_1, W_2, ….., W_n (see page 234)

Step 2-3 $u_1'=\frac{W_1}{W}$ $u_2'=\frac{W_2}{W}$ ……… $u_n'=\frac{W_n}{W}$

Step 2-4 $u_1=\int u_1'(x)dx$ $u_2=\int u_2'(x)dx$ ……. $u_n=\int u_n'(x)dx$

Step 2-5 $y_p(x)=u_1(x)y_1(x)+u_2(x)y_2(x)+\cdots\cdots+u_n(x)y_n(x)$

4-6-7 本節需注意的地方

(1) 養成先解 associated homogeneous equation 的習慣

(2) 記熟幾個重要公式

(3) 這裡 | | 指的是 determinant

(4) 算出 $u_1'(x)$ 和 $u_2'(x)$ 後別忘了作積分

特別要小心

(5) $f(x) = g(x)/a_n(x)$ (和 1st order 的情形一樣，使用 standard form)

(6) 計算 $u_1'(x)$ 和 $u_2'(x)$ 的積分時，$+ c$ 可忽略

因為我們的目的是算particular solution y_p, y_p 是任何一個能滿足原式的解

(7) 這方法解的範圍，不包含 $a_n(x) = 0$ 的地方

4-7 Cauchy-Euler Equation

4-7-1 解法限制條件

$$a_n x^n y^{(n)}(x) + a_{n-1} x^{n-1} y^{(n-1)}(x) + \cdots + a_1 x y'(x) + a_0 y = g(x)$$

not constant coefficients

k

but the coefficients of $y^{(k)}(x)$ have the form of $a_k x^k$

a_k is some constant

associated homogeneous equation

particular solution

$$a_n x^n y^{(n)}(x) + a_{n-1} x^{n-1} y^{(n-1)}(x) + \cdots + a_1 x y'(x) + a_0 y = 0$$

4-7-2 解法

Associated homogeneous equation of the Cauchy-Euler equation

$$a_n x^n y^{(n)}(x) + a_{n-1} x^{n-1} y^{(n-1)}(x) + \cdots + a_1 x y'(x) + a_0 y = 0$$

Guess the solution as $y(x) = x^m$, then

$$a_n x^n m(m-1)(m-2)\cdots\cdots(m-n+1)\, x^{m-n} +$$

$$a_{n-1} x^{n-1} m(m-1)(m-2)\cdots\cdots(m-n+2)\, x^{m-n+1} +$$

$$a_{n-2} x^{n-2} m(m-1)(m-2)\cdots\cdots(m-n+3)\, x^{m-n+2} +$$

$$\vdots$$

$$+a_1 x m x^{m-1}$$

$$+a_0 x^m = 0$$

$$
\begin{aligned}
& a_n m(m-1)(m-2)\cdots\cdots(m-n+1) \\
& +a_{n-1} m(m-1)(m-2)\cdots\cdots(m-n+2) \\
& +a_{n-2} m(m-1)(m-2)\cdots\cdots(m-n+3) \\
& \quad\vdots \\
& +a_1 m \\
& +a_0 = 0
\end{aligned}
$$

⟹ auxiliary function

比較: 和 constant coefficient 時有何不同？

規則：把 $x^k \dfrac{d^k}{dx^k}$ 變成 $\dfrac{m!}{(m-k)!}$

4-7-3 For the 2nd Order Case

$$a_2 x^2 y''(x) + a_1 x y'(x) + a_0 y = 0$$

auxiliary function:

$$a_2 m(m-1) + a_1 m + a_0 = 0 \qquad a_2 m^2 + (a_1 - a_2)m + a_0 = 0$$

roots

$$m_1 = \frac{a_2 - a_1 + \sqrt{(a_1 - a_2)^2 - 4a_2 a_0}}{2a_2} \qquad m_2 = \frac{a_2 - a_1 - \sqrt{(a_1 - a_2)^2 - 4a_2 a_0}}{2a_2}$$

[Case 1]: $m_1 \neq m_2$ and m_1, m_2 are real

two independent solution of the homogeneous part:

x^{m_1} and x^{m_2}

$$y_c = c_1 x^{m_1} + c_2 x^{m_2}$$

[Case 2]: $m_1 = m_2$

Use the method of reduction of order

$$y_1 = x^{m_1}$$

$$y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx = x^{m_1}\int \frac{e^{-\int \frac{a_1}{a_2 x}dx}}{x^{2m_1}}dx$$

Note 1: 原式 $\longrightarrow y''(x) + \frac{a_1}{a_2 x}y'(x) + \frac{a_0}{a_2 x^2}y = 0, \qquad P(x) = \frac{a_1}{a_2 x}$

Note 2: 此時 $m_1 = m_2 = \frac{a_2 - a_1}{2a_2}$

$$y_2(x) = x^{m_1}\int \frac{e^{-\int \frac{a_1}{a_2 x}dx}}{x^{2m_1}}dx = x^{m_1}\int \frac{e^{-\frac{a_1}{a_2}\ln|x|}}{x^{2m_1}}dx = x^{m_1}\int \frac{|x|^{-\frac{a_1}{a_2}}}{x^{2m_1}}dx$$

$$= (-1)^{-\frac{a_1}{a_2}} x^{m_1}\int x^{-\frac{a_1}{a_2}} x^{\frac{a_1-a_2}{a_2}} dx = x^{m_1}\int x^{-1}dx = x^{m_1}\ln|x|$$

If $y_2(x)$ is a solution of a homogeneous DE

then $c\,y_2(x)$ is also a solution of the homogeneous DE

If we constrain that $x > 0$, then $y_2 = x^{m_1}\ln x$

$$y_c = c_1 x^{m_1} + c_2 x^{m_2}\ln x$$

[Case 3]: $m_1 \neq m_2$ and m_1, m_2 are the form of

$$m_1 = \alpha + j\beta \qquad m_2 = \alpha - j\beta$$

two independent solution of the homogeneous part:

$$x^{\alpha + j\beta} \quad \text{and} \quad x^{\alpha - j\beta}$$

$$y_c = C_1 x^{\alpha + j\beta} + C_2 x^{\alpha - j\beta}$$

$$x^{\alpha + j\beta} = (e^{\ln x})^{\alpha + j\beta} = e^{(\alpha + j\beta)\ln x} = e^{\alpha \ln x} e^{j\beta \ln x}$$

$$= x^{\alpha}\left(\cos(\beta \ln x) + j\sin(\beta \ln x)\right)$$

同理 $x^{\alpha - j\beta} = x^{\alpha}\left(\cos(\beta \ln x) - j\sin(\beta \ln x)\right)$

$$y_c = x^{\alpha}[(C_1 + C_2)\cos(\beta \ln x) + (C_1 - C_2)\sin(\beta \ln x)]$$

$$\boxed{y_c = x^{\alpha}[c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)]}$$

Example 1 (text page 164)

$$x^2 y''(x) - 2xy'(x) - 4y = 0$$

Example 2 (text page 164)

$$4x^2 y''(x) + 8xy'(x) + y = 0$$

Example 3 (text page 165)

$$4x^2 y''(x) + 17y = 0 \qquad y(1) = -1 \qquad y'(1) = -\frac{1}{2}$$

4-7-4 For the Higher Order Case

Process: auxiliary function

↓

roots — Step 1-1

↓

independent solutions — Step 1-2

↓

solution of the associated homogeneous equation — Step 1-3

(1) 若 auxiliary function 在 m_0 的地方只有一個根

x^{m_0}

是 associated homogeneous equation 的其中一個解

(2) 若 auxiliary function 在 m_0 的地方有 k 個重根

x^{m_0} , $x^{m_0}\ln x$, $x^{m_0}(\ln x)^2$, ... , $x^{m_0}(\ln x)^{k-1}$

皆為 associated homogeneous equation 的解

(3) 若 auxiliary function 在 $\alpha + j\beta$ 和 $\alpha - j\beta$ 的地方各有一個根 (未出現重根)

$$x^{\alpha}\cos(\beta\ln x),\quad x^{\alpha}\sin(\beta\ln x)$$

是 associated homogeneous equation 的其中二個解

(4) 若 auxiliary function 在 $\alpha + j\beta$ 和 $\alpha - j\beta$ 的地方皆有 k 個重根

$$x^{\alpha}\cos(\beta\ln x),\quad x^{\alpha}\cos(\beta\ln x)\ln x,\quad x^{\alpha}\cos(\beta\ln x)(\ln x)^{2},\quad \cdots\cdots,$$
$$x^{\alpha}\cos(\beta\ln x)(\ln x)^{k-1}$$
$$x^{\alpha}\sin(\beta\ln x),\quad x^{\alpha}\sin(\beta\ln x)\ln x,\quad x^{\alpha}\sin(\beta\ln x)(\ln x)^{2},\quad \cdots\cdots,$$
$$x^{\alpha}\sin(\beta\ln x)(\ln x)^{k-1}$$

是 associated homogeneous equation 的其中 $2k$ 個解

Example 4 (text page 166)

$$x^3 y'''(x) + 5x^2 y''(x) + 7xy'(x) + 8y = 0$$

auxiliary function

$$m(m-1)(m-2) + 5m(m-1) + 7m + 8 = 0$$

$$m^3 - 3m^2 + 2m + 5m^2 - 5m + 7m + 8 = 0$$

$$m^3 + 2m^2 + 4m + 8 = 0$$

$$(m+2)\left(m^2 + 4\right) = 0$$

4-7-5 Nonhomogeneous Case

To solve the nonhomogeneous Cauchy-Euler equation:

Method 1: (See Example 5)

(1) Find the complementary function (general solutions of the associated homogeneous equation) from the rules on pages 241-248.

(2) Use the method in Sec.4-6 (Variation of Parameters) to find the particular solution.

(3) Solution = complementary function + particular solution

Method 2: See Example 6，重要

Set $x = e^t$, $t = \ln x$

Example 5 (text page 166, illustration for method 1)

$$x^2 y''(x) - 3xy'(x) + 3y = 2x^4 e^x$$

Step 1 solution of the associated homogeneous equation

auxiliary function

$$m(m-1) - 3m + 3 = 0 \qquad m^2 - 4m + 3 = 0 \qquad m_1 = 1$$

$$y_c = c_1 x + c_2 x^3 \qquad m_2 = 3$$

Step 2-2 Particular solution $W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 2x^3$

$$W_1 = \begin{vmatrix} 0 & x^3 \\ 2x^2 e^x & 3x^2 \end{vmatrix} = -2x^5 e^x \qquad W_2 = \begin{vmatrix} x & 0 \\ 1 & 2x^2 e^x \end{vmatrix} = 2x^3 e^x$$

Step 2-3 $u_1' = \frac{W_1}{W} = -x^2 e^x \qquad u_2' = \frac{W_2}{W} = e^x$

Step 2-4 $u_1 = \int u_1' dx = -x^2 e^x + 2xe^x - 2e^x$

$$u_2 = \int u_2' dx = e^x$$

Step 2-5 $y_p = u_1 y_1 + u_2 y_2 = 2x^2 e^x - 2xe^x$

Step 3 $y = c_1 x + c_2 x^3 + 2x^2 e^x - 2xe^x$

Example 6 (text page 167, illustration for method 2)

$$x^2 y''(x) - xy'(x) + y = \ln x$$

Set $x = e^t$, $t = \ln x$

$$\frac{dy}{dx} = \frac{dt}{dx}\frac{dy}{dt} = \frac{1}{x}\frac{dy}{dt} \qquad \text{(chain rule)}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dt}{dx}\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{1}{x}\frac{d}{dt}\left(\frac{1}{x}\frac{dy}{dt}\right)$$

$$= \frac{1}{x^2}\frac{d^2y}{dt^2} + \frac{1}{x}\left(\frac{d}{dt}\frac{1}{x}\right)\left(\frac{dy}{dt}\right) = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

Therefore, the original equation is changed into

$$\frac{d^2}{dt^2}y(t) - 2\frac{d}{dt}y(t) + y(t) = t$$

$$\frac{d^2}{dt^2}y(t)-2\frac{d}{dt}y(t)+y(t)=t$$

$$\Longrightarrow \quad y(t)=c_1e^t+c_2te^t+t+2$$

$$\Longrightarrow \quad y(x)=c_1x+c_2x\ln x+\ln x+2$$

(別忘了 $t=\ln x$ 要代回來)

Note 1: 以此類推

$$\frac{d^k y}{dx^k}=\frac{1}{x^k}(D_t-k+1)\cdots(D_t-1)D_t y$$

D_t means $\dfrac{d}{dt}$

Note 2: 簡化計算的小技巧：結合兩種解 nonhomogeneous Cauchy-Euler equation 的長處

4-7-6 本節要注意的地方

(1) 本節公式記憶的方法：

把 Section 4-3 的 e^x 改成 x，x 改成 $\ln(x)$

把 auxiliary function 的 m^n 改成 $m(m-1)(m-2)\cdots\cdots(m-n+1)$

(2) 如何解 particular solution?

Variation of Parameters 的方法

(3) 解的範圍將不包括 $x = 0$ 的地方 (Why?)

還有很多 linear DE 沒有辦法解，怎麼辦

(1) numerical approach (Section 4-9-3)

(2) using special function (Chap. 6)

(3) Laplace transform and Fourier transform (Chaps. 7, 11, 14)

(4) 查表 (table lookup)

(1) 即使用了 Section 4-7 的方法，大部分的 DE還是沒有辦法解

(2) 所幸，自然界真的有不少的例子是 linear DE

甚至是 constant coefficient linear DE

Exercise for practice

Section 4-4	5, 6, 14, 17, 18, 24, 26, 33, 39, 42
Section 4-5	2, 7, 13, 18, 31, 45, 69, 70
Section 4-6	4, 5, 8, 13, 14, 18, 21, 24, 25, 28
Section 4-7	11, 17, 18, 20, 21, 32, 34, 36, 37
Review 4	2, 13, 14, 17, 19, 20, 23, 24, 27, 32

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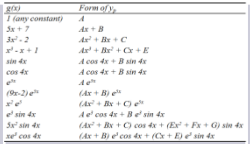
<tr><td>191</td><td>

$g(x)$	Form of y_p
1 (any constant)	A
$5x + 7$	$Ax + B$
$3x^2 - 2$	$Ax^2 + Bx + C$
$x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
$\sin 4x$	$A \cos 4x + B \sin 4x$
$\cos 4x$	$A \cos 4x + B \sin 4x$
e^{5x}	$A e^{5x}$
$(9x-2) e^{5x}$	$(Ax + B) e^{5x}$
$x^2 e^5$	$(Ax^2 + Bx + C) e^{5x}$
$e^3 \sin 4x$	$A e^3 \cos 4x + B e^3 \sin 4x$
$5x^2 \sin 4x$	$(Ax^2 + Bx + C) \cos 4x + (Ex^2 + Fx + G) \sin 4x$
$xe^3 \cos 4x$	$(Ax + B) e^3 \cos 4x + (Cx + E) e^3 \sin 4x$

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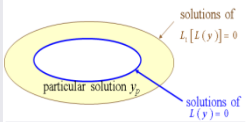
<tr><td>217</td><td>
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