

工程數學--微分方程

Differential Equations (DE)

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教學網頁：<http://djj.ee.ntu.edu.tw/DE.htm>



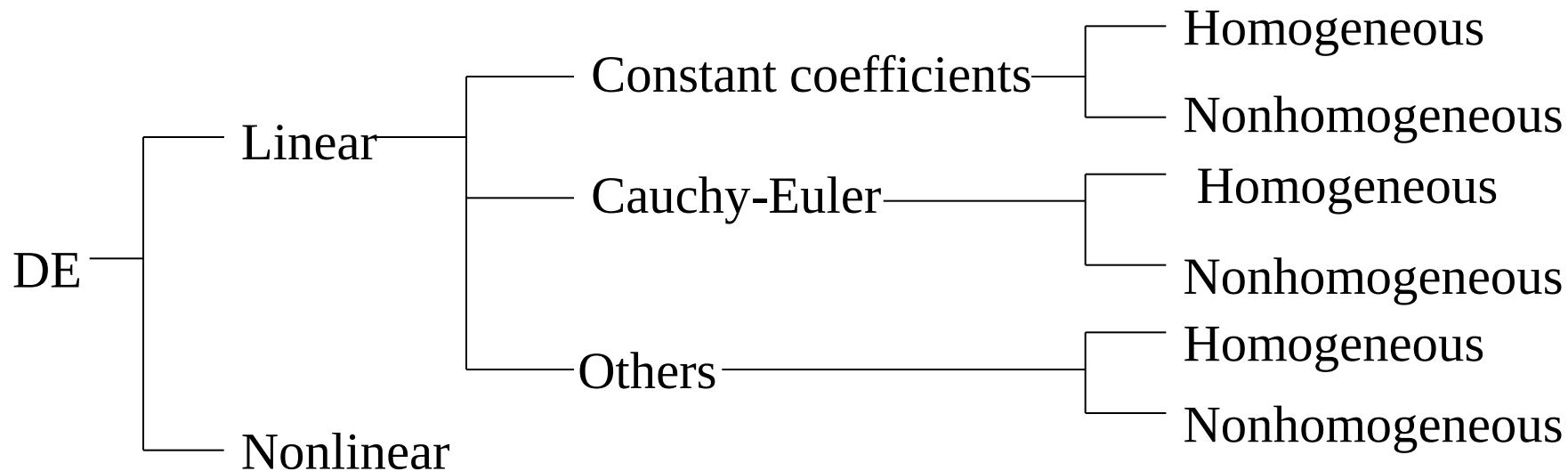
【本著作除另有註明外，採取創用CC「姓名標示—非商業性—相同方式分享」台灣3.0版授權釋出】

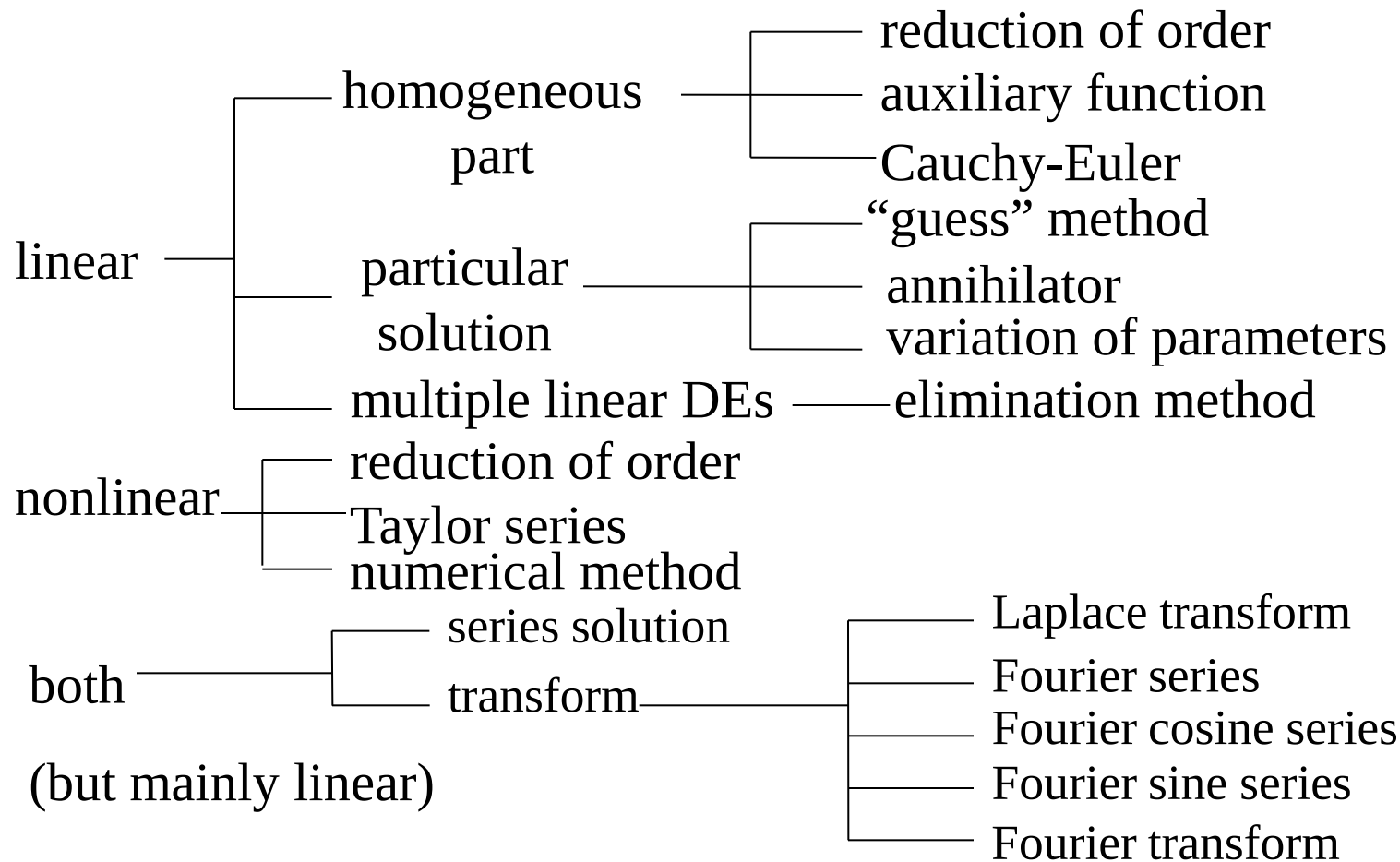
Chapter 4 Higher Order Differential Equations

Highest differentiation: $\frac{d^n y}{dx^n}$, $n > 1$

Most of the methods in Chapter 4 are applied for the **linear** DE.

附錄三 DE 的分類





4-1 Linear Differential Equations: Basic Theory

4.1.1 Initial-Value and Boundary Value Problems

4.1.1.1 n^{th} order Initial Value Problem

i.e., n^{th} order linear DE with IVP at the same point

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x) y = g(x)$$

$$y(x_0) = y_0 \quad y'(x_0) = y_1 \quad y''(x_0) = y_2$$

$$\dots\dots\dots y^{(n-1)}(x_0) = y_{n-1}$$

initial condition

For an interval I that contains the point x_0

- ① If $a_0(x), a_1(x), a_2(x), \dots, a_{n-1}(x), a_n(x)$ are continuous at $x = x_0$
- ② $a_n(x_0) \neq 0$

(很像沒有 singular point 的條件)

then for the problem on page 136, the solution $y(x)$ exists and is unique on the interval I that contains the point x_0

(Interval I 的範圍，取決於何時 $a_n(x) = 0$ 以及 何時 $a_k(x)$ ($k = 0 \sim n$) 不為continuous)

Otherwise, the solution is either **non-unique** or does **not exist**.

(infinite number of solutions) (no solution)

Example 1 (text page 118)

$$3y''' + 5y'' - y' + 7y = 0 \quad y(1) = 0 \quad y'(1) = 0 \quad y''(1) = 0$$

Example 2 (text page 119)

$$y'' - 4y' = 12x \quad y(0) = 4 \quad y'(0) = 1$$

- $x^2 y'' - 2xy' + 2y = 6 \quad y(0) = 3 \quad y'(0) = 1$

有無限多組解

$$y = cx^2 + x + 3 \quad c \text{ 為任意之常數}$$

- 比較：

$$x^2 y'' - 2xy' + 2y = 6 \quad y(1) = 3 \quad y'(1) = 1$$

只有一個解

$$y = x^2 - x + 3$$

$$x \in (0, \infty)$$

- Note:

The initial value can also be the form as:

$$\alpha y(x_0) + \beta y'(x_0) = 0$$

$$\sum_{n=0}^{N-1} \alpha_n y^{(n)}(x_0) = 0 \quad (\text{general initial condition})$$

4.1.1.2 n^{th} Order Boundary Value Problem

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Boundary conditions are specified at different points

比較：Initial conditions are specified at the same points

例子： $a_2(x)y'' + a_1(x)y' + a_0(x) = g(x)$

subject to $y(a) = y_0, \quad y(b) = y_1$

或 $y'(a) = y_0, \quad y(b) = y_1$

或
$$\begin{cases} \alpha_1 y(a) + \beta_1 y'(a) = \gamma_1 \\ \alpha_2 y(b) + \beta_2 y'(b) = \gamma_2 \end{cases}$$

An n^{th} order linear DE with n boundary conditions may have a unique solution, no solution, or infinite number of solutions.

Example 3 (text page 120)

$$y'' + 16y = 0$$

solution: $y = c_1 \cos(4x) + c_2 \sin(4x)$

$$(1) \quad y(0) = 0 \quad y(\pi/2) = 0$$

$y = c_2 \sin(4x)$ c_2 is any constant (infinite number of solutions)

$$(2) \quad y(0) = 0 \quad y(\pi/8) = 0$$

$y = 0$ (unique solution)

4.1.2 Homogeneous Equations

4.1.2.1 Definition

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x) y = g(x)$$

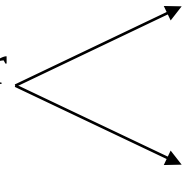
$g(x) = 0 \longrightarrow$ homogeneous

$g(x) \neq 0 \longrightarrow$ nonhomogeneous

- 重要名詞：Associated homogeneous equation

The associated homogeneous equation of a nonhomogeneous DE:

Setting $g(x) = 0$

- Review: solving the 1st order
nonhomogeneous
linear DE
- 

4.1.2.2 New Notations

Notation: $D^n y = \frac{d^n y}{dx^n}$

$$\frac{d^2 y}{dx^2} + 5 \frac{dy}{dx} + 6y \xrightarrow{\text{可改寫成}} D^2 y + 5Dy + 6y \xrightarrow{\text{可改寫成}} (D^2 + 5D + 6)y$$

可再改寫成

$$L(y)$$
$$L = D^2 + 5D + 6$$

4.1.2.3 Solution of the Homogeneous Equation

[Theorem 4.1.5]

For an n^{th} order homogeneous linear DE $L(y) = 0$, if

- ① $y_1(t), y_2(t), \dots, y_n(t)$ are the solutions of $L(y) = 0$
- ② $y_1(t), y_2(t), \dots, y_n(t)$ are linearly independent

then any solution of the homogeneous linear DE can be expressed as:

$$y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$$

可以和矩陣的概念相比較

From Theorem 4.1.5:

An n^{th} order homogeneous linear DE has n linearly independent solutions.

Find n linearly independent solutions

== Find all the solutions of an n^{th} order homogeneous linear DE

$y_1(t), y_2(t), \dots, y_n(t)$: fundamental set of solutions

$y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$: general solution of the homogenous linear DE

(又稱做 complementary function)

也是重要名詞

Definition 4.1 Linear Dependence / Independence

If there is no solution other than $c_1 = c_2 = \dots = c_n = 0$ for the following equality

$$c_1 y_1(x) + c_2 y_2(x) + \dots + c_n y_n(x) = 0$$

then $y_1(t), y_2(t), \dots, y_n(t)$ are said to be **linearly independent**.

Otherwise, they are **linearly dependent**.

判斷是否為 linearly independent 的方法: **Wronskian**

Definition 4.2 Wronskian

$$W(y_1, y_2, \dots, y_n) = \det \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix}$$

$W(y_1, y_2, \dots, y_n) \neq 0 \longrightarrow$ linearly independent

4.1.2.4 Examples

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Example 9 (text page 125)

$$y''' - 6y'' + 11y' - 6y = 0$$

$y_1 = e^x$, $y_2 = e^{2x}$, and $y_3 = e^{3x}$ are three of the solutions

Since

$$\det \begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix} = \begin{bmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{bmatrix} = e^{x+2x+3x} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} = 2e^{6x} \neq 0$$

Therefore, y_1 , y_2 , and y_3 are linear independent for any x
general solution:

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x} \quad x \in (-\infty, \infty)$$

4.1.3 Nonhomogeneous Equations (可和 page 53 相比較)

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Nonhomogeneous **linear** DE

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y = g(x)$$

Part 1

Associated homogeneous DE

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y = 0$$

find n linearly independent solutions

$$y_1(x), y_2(x), \dots, y_n(x)$$

Part 2

particular solution y_p

(any solution of the nonhomogeneous linear DE)

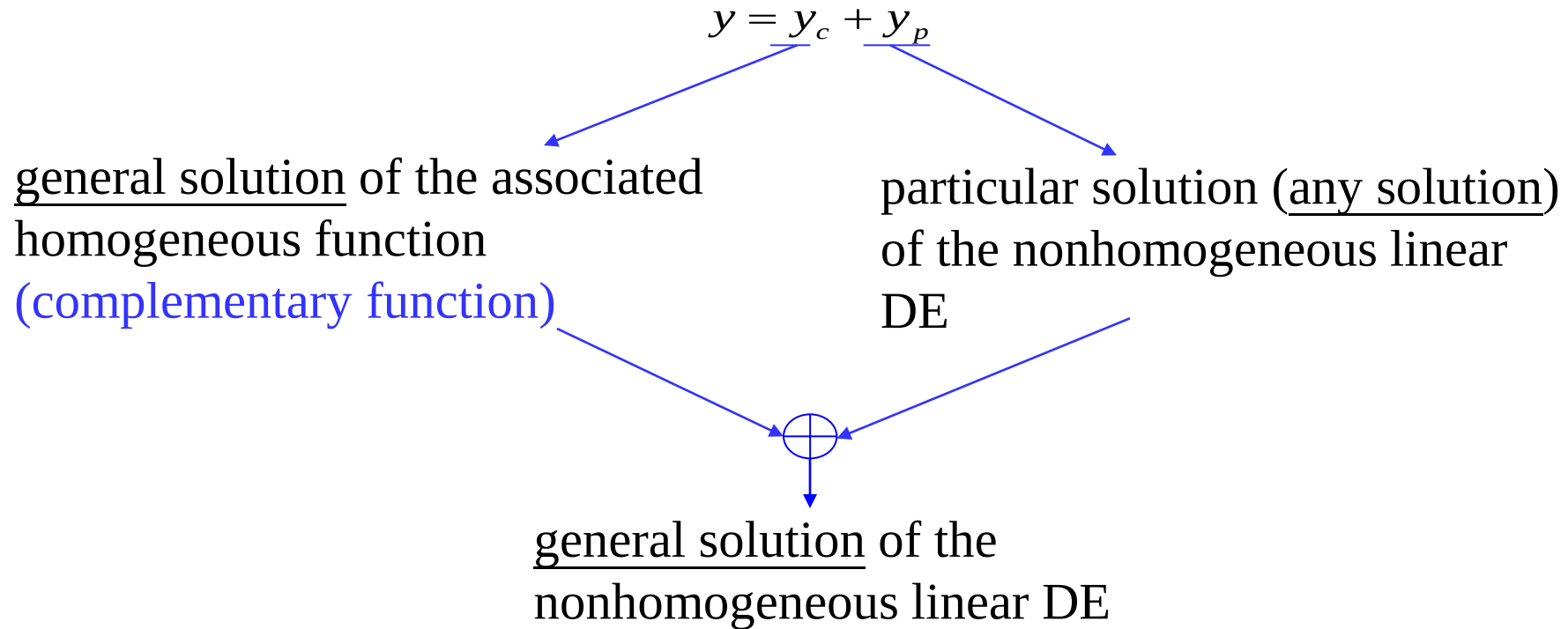
$$g(x) = g_1(x) + g_2(x) + \cdots + g_k(x)$$

$$y_p(x) = \underset{\downarrow}{y_{p_1}}(x) + \underset{\downarrow}{y_{p_2}}(x) + \cdots + \underset{\downarrow}{y_{p_k}}(x)$$

general solution of the nonhomogeneous linear DE

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x)$$

Theorem 4.1.6 general solution of a nonhomogeneous linear DE



Example 10 (text page 126)

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$$y''' - 6y'' + 11y' - 6y = 3x$$

$$y''' - 6y'' + 11y' - 6y = 0$$

Three linearly independent solution

$$e^x, e^{2x}, e^{3x}$$

Check by Wronskian (Example 9)

$$\begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = 2e^{6x}$$

Particular solution

$$y_p = -\frac{11}{12} - \frac{1}{2}x$$

General solution:

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x} - \frac{11}{12} - \frac{1}{2}x$$

Theorem 4.1.7 Superposition Principle

If $y_{p_1}(x)$ is the particular solution of

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = g_1(x)$$

$y_{p_2}(x)$ is the particular solution of

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = g_2(x)$$

$y_{p_k}(x)$ is the particular solution of

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = g_k(x)$$

then $y_{p_1}(x) + y_{p_2}(x) + \cdots + y_{p_k}(x)$ is the particular solution of

$$\begin{aligned} & a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) \\ &= g_1(x) + g_2(x) + \cdots + g_k(x) \end{aligned}$$

Example 11 (text page 127)

$y_{p_1}(x) = -4x^2$ is a particular solution of $y'' - 3y' + 4y = -16x^2 + 24x - 8$

$y_{p_2}(x) = e^{2x}$ is a particular solution of $y'' - 3y' + 4y = 2e^{2x}$

$y_{p_3}(x) = xe^x$ is a particular solution of $y'' - 3y' + 4y = 2xe^x - e^x$

$y = y_{p_1} + y_{p_2} + y_{p_3} = -4x^2 + e^{2x} + xe^x$ is a particular solution of

$$y'' - 3y' + 4y = -16x^2 + 24x - 8 + 2e^{2x} + 2xe^x - e^x$$

4.1.4 名詞

- initial conditions, boundary conditions (pages 136, 140)
- associated homogeneous equation , complementary function (page 142)
- fundamental set of solutions (page 145)
- Wronskian (page 147)
- particular solution (page 149)
- general solution of the homogenous linear DE (page 145)
- general solution of the nonhomogenous linear DE (page 149)

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) = g(t)$$

4.1.5 本節要注意的地方

- (1) Most of the theories in Section 4.1 are applied to the linear DE
- (2) 注意 initial conditions 和 boundary conditions 之間的不同
- (3) 快速判斷 linear independent

(補充 1) Theorem 4.1.1 的解釋

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x) y = g(x)$$

$$y(x_0) = y_0 \quad y'(x_0) = y_1 \quad \dots \dots \dots y^{(n-1)}(x_0) = y_{n-1}$$

When $a_n(x_0) \neq 0$

$$y^{(n)}(x_0) + \frac{a_{n-1}(x_0)}{a_n(x_0)} y^{(n-1)}(x_0) + \cdots + \frac{a_1(x_0)}{a_n(x_0)} y'(x_0) + \frac{a_0(x_0)}{a_n(x_0)} y(x_0) = \frac{g(x_0)}{a_n(x_0)}$$

find $y^{(n)}(x_0)$

$$y^{(n-1)}(x_0 + \Delta) = y^{(n-1)}(x_0) + y^{(n)}(x_0) \Delta \longrightarrow \text{find } y^{(n-1)}(x_0 + \Delta)$$

(根據 $f'(t) = \frac{f(t+\Delta) - f(t)}{\Delta}$, $f(t+\Delta) = f(t) + f'(t)\Delta$)

以此類推

$$y^{(n-2)}(x_0 + \Delta) = y^{(n-2)}(x_0) + y^{(n-1)}(x_0)\Delta \longrightarrow \text{find } y^{(n-2)}(x_0 + \Delta)$$

$$y^{(n-3)}(x_0 + \Delta) = y^{(n-3)}(x_0) + y^{(n-2)}(x_0)\Delta \longrightarrow \text{find } y^{(n-3)}(x_0 + \Delta)$$

$$\vdots$$

$$\vdots$$

$$y(x_0 + \Delta) = y(x_0) + y'(x_0)\Delta \longrightarrow \boxed{\text{find } y(x_0 + \Delta)}$$



$$y^{(n)}(x_0 + \Delta) + \frac{a_{n-1}(x_0 + \Delta)}{a_n(x_0 + \Delta)} y^{(n-1)}(x_0 + \Delta) + \cdots + \frac{a_1(x_0 + \Delta)}{a_n(x_0 + \Delta)} y'(x_0 + \Delta) + \frac{a_0(x_0 + \Delta)}{a_n(x_0 + \Delta)} y(x_0 + \Delta) = \frac{g(x_0 + \Delta)}{a_n(x_0 + \Delta)} \longrightarrow \text{find } y^{(n)}(x_0 + \Delta)$$



$$y^{(n-1)}(x_0 + 2\Delta) = y^{(n-1)}(x_0 + \Delta) + y^{(n)}(x_0 + \Delta)\Delta \longrightarrow \text{find } y^{(n-1)}(x_0 + 2\Delta)$$

$$y^{(n-2)}(x_0 + 2\Delta) = y^{(n-2)}(x_0 + \Delta) + y^{(n-1)}(x_0 + \Delta)\Delta \longrightarrow \text{find } y^{(n-2)}(x_0 + 2\Delta)$$

$$\begin{array}{c}
 \vdots \\
 y(x_0 + 2\Delta) = y(x_0 + \Delta) + y'(x_0 + \Delta)\Delta \quad \longrightarrow \quad \boxed{\text{find } y(x_0 + 2\Delta)}
 \end{array}$$

$$\begin{aligned}
 & y^{(n)}(x_0 + 2\Delta) + \frac{a_{n-1}(x_0 + 2\Delta)}{a_n(x_0 + 2\Delta)} y^{(n-1)}(x_0 + 2\Delta) + \cdots + \frac{a_1(x_0 + 2\Delta)}{a_n(x_0 + 2\Delta)} y'(x_0 + 2\Delta) \\
 & + \frac{a_0(x_0 + 2\Delta)}{a_n(x_0 + 2\Delta)} y(x_0 + 2\Delta) = \frac{g(x_0 + 2\Delta)}{a_n(x_0 + 2\Delta)}
 \end{aligned}$$

以此類推，可將 $y(x_0 + 3\Delta)$, $y(x_0 + 4\Delta)$, $y(x_0 + 5\Delta)$,

以至於將 $y(x)$ 所有的值都找出來。

(求 $y(x)$ for $x > x_0$ 時, 用正的 Δ 值，

求 $y(x)$ for $x < x_0$ 時, 用負的 Δ 值)

Requirement 1: $a_0(x), a_1(x), a_2(x), \dots, a_{n-1}(x), a_n(x)$ are continuous
是為了讓 $a_k(x_0+m\Delta)$ 皆可以定義

Requirement 2: $a_n(x) \neq 0$ 是為了讓 $a_k(x_0+m\Delta) / a_n(x_0+m\Delta)$ 不為無限大

4-2 Reduction of Order

4.2.1 適用情形

(1) (2) (3)

Suitable for the 2nd order linear homogeneous DE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

(4) One of the nontrivial solution $y_1(x)$ has been known.

4.2.2 解法

假設 $y_2(x) = u(x)y_1(x)$

先將DE 變成 Standard form

$$y'' + P(x)y' + Q(x)y = 0$$

If $y(x) = u(x)y_1(x)$,

$$y' = uy_1' + u'y_1 \qquad y'' = uy_1'' + 2u'y_1' + u''y_1$$

$$uy_1'' + 2u'y_1' + u''y_1 + P(x)uy_1' + P(x)u'y_1 + Q(x)uy_1 = 0$$

$$u(\underbrace{y_1'' + P(x)y_1' + Q(x)y_1}_{\text{zero}}) + 2u'y_1' + u''y_1 + P(x)u'y_1 = 0$$

zero

$$u''y_1 + u'(2y_1' + P(x)y_1) = 0$$

set $w = u'$

$$\frac{dw}{dx}y_1 + w(2\frac{dy_1}{dx} + P(x)y_1) = 0$$

multiplied by $dx/(y_1w)$

$$\frac{dw}{w} + 2\frac{dy_1}{y_1} + P(x)dx = 0$$

separable variable
(with 3 variables)

$$\int \frac{dw}{w} + 2\int \frac{dy_1}{y_1} + \int P(x)dx = 0$$

$$\ln|w| + c_3 + 2\ln|y_1| + c_4 = -\int P(x)dx$$

$$\ln|w| + 2\ln|y_1| = \ln|w| + \ln|y_1|^2 = \ln|w||y_1|^2 = \ln|wy_1^2|$$

$$\ln|wy_1^2| = -\int P(x)dx + c$$

$$\ln |wy_1^2| = -\int P(x) dx + c$$

$$wy_1^2 = \pm e^{-\int P(x) dx + c}$$

$$w = c_1 e^{-\int P(x) dx} / y_1^2$$

$$u = \int w dx = c_1 \int \frac{e^{-\int P(x) dx}}{y_1^2} dx + c_2$$

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x) dx}}{y_1^2(x)} dx$$

We can setting $c_1 = 1$ and $c_2 = 0$

(因為我們算 $u(x)$ 的目的，只是為了要算出與 $y_1(x)$ 互相independent的另一個解)

4.2.3 例子

Example 1 (text page 131)

$$y'' - y = 0$$

We have known that $y_1 = e^x$ is one of the solution

$$P(x) = 0 \quad y_2(x) = e^x \int c e^{-2x} dx = -\frac{1}{2} c e^{-x}$$

Specially, set $c = -2$, ($y_2(x)$ 只要 independent of $y_1(x)$ 即可
所以 c 的值可以任意設)

$$y_2(x) = e^{-x}$$

General solution:

$$y(x) = c_1 e^x + c_2 e^{-x}$$

Example 2 (text page 132)

(將課本 x 的範圍做更改)

$$x^2 y'' - 3xy' + 4y = 0 \quad \text{when } x \in (-\infty, 0)$$

We have known that $y_1 = x^2$ is one of the solution

Note: the interval of x

If $x \in (0, \infty)$ ($x > 0$), $\int dx/x = \ln x$ 如課本

If $x < 0$, $\int dx/x = \ln(-x)$

$$\begin{aligned} y_2(x) &= x^2 \int \frac{e^{3\ln(-x)}}{x^4} dx = x^2 \int \frac{(-x)^3}{(-x)^4} dx \\ &= -x^2 \int \frac{1}{x} dx = -x^2 \ln|x| \end{aligned}$$

$$y(x) = c_1 x^2 + c_2 x^2 \ln|x|$$

4.2.4 本節需注意的地方

- (1) 記住公式
$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx$$
- (2) 若不背公式 (不建議)，在計算過程中別忘了對 $w(x)$ 做積分
- (3) 別忘了 $P(x)$ 是 “standard form” 一次微分項的 coefficient term
- (4) 同樣有 singular point 的問題
- (5) 因為 $y_2(x)$ 是 homogeneous linear DE 的 “任意解”，所以計算時，常數的選擇以方便為原則
- (6) 由於 $\int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx$ 的計算較複雜且花時間，所以要多加練習

多算習題

附錄五： Hyperbolic Function

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

比較： $\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}$

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

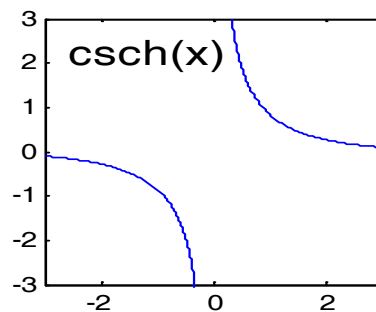
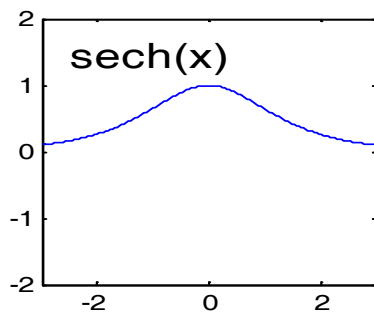
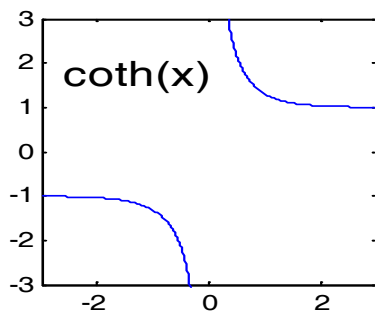
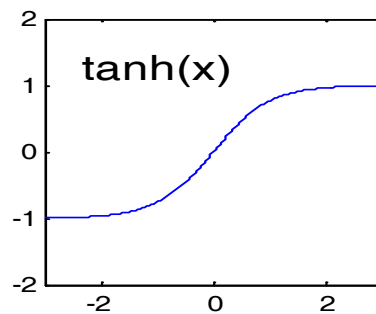
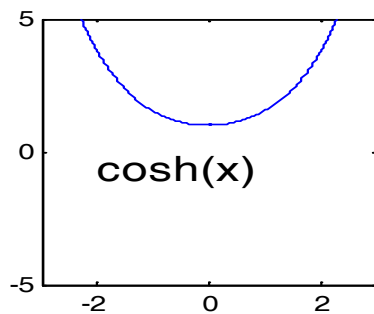
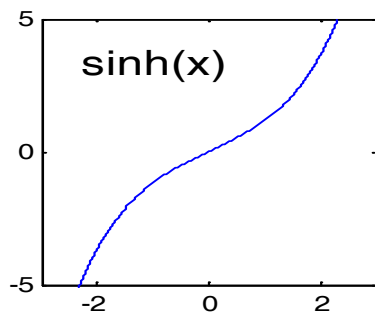
$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^x + e^{-x}}$$

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)} = \frac{2}{e^x - e^{-x}}$$



$$\frac{d}{dx} \sinh(ax) = a \cosh(ax)$$

$$\sinh(0) = 0$$

$$\frac{d}{dx} \cosh(ax) = a \sinh(ax)$$

$$\cosh(0) = 1$$

$$\frac{d}{dx} \tanh(ax) = a \operatorname{sech}^2(ax)$$

$$\sinh'(0) = 1$$

$$\frac{d}{dx} \coth(ax) = -a \operatorname{csch}^2(ax)$$

$$\cosh'(0) = 0$$

$$\frac{d}{dx} \operatorname{sech}(ax) = -a \operatorname{sech}(ax) \tanh(ax)$$

$$\sin(ix) = i \sinh(x)$$

$$\frac{d}{dx} \operatorname{csch}(ax) = -a \operatorname{csch}(ax) \coth(ax)$$

$$\cos(ix) = \cosh(x)$$

$$\int \sinh(ax) dx = \frac{\cosh(ax)}{a} + c$$

$$\int \cosh(ax) dx = \frac{\sinh(ax)}{a} + c$$

$$\int \tanh(ax) dx = \frac{\ln|\cosh(ax)|}{a} + c$$

$$\int \coth(ax) dx = \frac{\ln|\sinh(ax)|}{a} + c$$

$$\int \operatorname{sech}(ax) dx = \frac{2 \tan^{-1}\left(\tanh\left(\frac{a}{2}x\right)\right)}{a} + c$$

$$\int \operatorname{csch}(ax) dx = \frac{\ln\left|\tanh\left(\frac{a}{2}x\right)\right|}{a} + c$$

附錄六 Linear DE 解法的步驟 (參照講義 page 149)

Step 1: Find the **general solution** (i.e., the **complementary function**)
of the **associated homogeneous DE**
(Sections 4-2, 4-3, 4-7)

Step 2: Find the **particular solution**
(Sections 4-4, 4-5, 4-6)

Step 3: Combine the complementary function and the particular solution

Extra Step: Consider the initial (or boundary) conditions

4-3 Homogeneous Linear Equations with Constant Coefficients

本節使用 **auxiliary equation** 的方法來解 homogeneous DE

KK: [əg`zɪ lɪ ər i]

4-3-1 限制條件

限制條件: (1) homogeneous

(2) linear

(3) **constant coefficients**

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y = 0$$

$a_0, a_1, a_2, \dots, a_n$ are constants

(the simplest case of the higher order DEs)

4-3-2 解法

解法核心：

Suppose that the solutions has the form of e^{mx}

Example: $y''(x) - 3y'(x) + 2y(x) = 0$

Set $y(x) = e^{mx}$, $m^2 e^{mx} - 3m e^{mx} + 2 e^{mx} = 0$

$$m^2 - 3m + 2 = 0 \longrightarrow \text{solve } m$$

可以直接把 n 次微分 用 m^n 取代，變成一個多項式

這個多項式被稱為 auxiliary equation

• 解法流程

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y = 0$$

Step 1-1

auxiliary function

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0$$

Step 1-1

Find n roots, $m_1, m_2, m_3, \dots, m_n$

(If $m_1, m_2, m_3, \dots, m_n$ are distinct)

Step 1-2

n linearly independent solutions $e^{m_1 x}, e^{m_2 x}, e^{m_3 x}, \dots, e^{m_n x}$
(有三個 Cases)

Step 1-3

Complementary
function

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$

4-3-3 Three Cases for Roots (2nd Order DE)

$$a_2 y''(x) + a_1 y'(x) + a_0 y(x) = 0$$



$$a_2 m^2 + a_1 m + a_0 = 0$$

$$\text{roots } m_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2 a_0}}{2a_2} \quad m_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}$$

solutions

Case 1 $m_1 \neq m_2$, m_1, m_2 are real

(其實 m_1, m_2 不必限制為 real)

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

Case 2 $m_1 = m_2$ (m_1 and m_2 are of course real)

First solution: $y_1 = e^{m_1 x}$

Second solution: using the method of “Reduction of Order”

$$\begin{aligned}
 y_2(x) &= y_1(x) \int \frac{e^{-\int P(x) dx}}{y_1^2(x)} dx \\
 &= e^{m_1 x} \int e^{-2m_1 x} e^{-\int a_1/a_2 dx} dx \\
 &= e^{m_1 x} \int e^{(-2m_1 - \frac{a_1}{a_2})x} dx \\
 &= e^{m_1 x} \int dx = e^{m_1 x} (x + c)
 \end{aligned}
 \qquad m_1 = \frac{-a_1}{2a_2}$$

$$y_2(x) = xe^{m_1 x}$$

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$$

Case 3 $m_1 \neq m_2$, m_1 and m_2 are conjugate and complex

$$m_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2a_0}}{2a_2} = \alpha + j\beta \quad m_2 = \alpha - j\beta$$

$$\alpha = -a_1 / 2a_2, \quad \beta = \sqrt{4a_2a_0 - a_1^2} / 2a_2$$

Solution: $y = C_1 e^{\alpha x + j\beta x} + C_2 e^{\alpha x - j\beta x}$

Another form:

$$\begin{aligned} y &= e^{\alpha x} (C_1 e^{j\beta x} + C_2 e^{-j\beta x}) \\ &= e^{\alpha x} (C_1 \cos \beta x + jC_1 \sin \beta x + C_2 \cos \beta x - jC_2 \sin \beta x) \end{aligned}$$

$$\text{set } c_1 = C_1 + C_2 \text{ and } c_2 = jC_1 - jC_2$$

$$y = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x) \quad c_1 \text{ and } c_2 \text{ are some constant}$$

Example 1 (text page 135)

(a) $2y'' - 5y' - 3y = 0$

$$2m^2 - 5m - 3 = 0, \quad m_1 = -1/2, \quad m_2 = 3$$

$$y = c_1 e^{-x/2} + c_2 e^{3x}$$

(b) $y'' - 10y' + 25y = 0$

$$m^2 - 10m + 25 = 0, \quad m_1 = 5, \quad m_2 = 5$$

$$y = c_1 e^{5x} + c_2 x e^{5x}$$

(c) $y'' + 4y' + 7y = 0$

$$m^2 + 4m + 7 = 0, \quad m_1 = -2 + i\sqrt{3}, \quad m_2 = -2 - i\sqrt{3}$$

$$y = e^{-2x} (c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x)$$

4-3-4 Three + 1 Cases for Roots (Higher Order DE)

For higher order case $a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y = 0$

auxiliary function: $a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0$

roots: $m_1, m_2, m_3, \dots, m_n$

(1) If $m_p \neq m_q$ for $p = 1, 2, \dots, n$ and $p \neq q$

(也就是這個多項式在 m_q 的地方只有一個根)

then $e^{m_q x}$ is a solution of the DE.

重覆次數

(2) If the multiplicities of m_q is k (當這個多項式在 m_q 的地方有 k 個根),

$$e^{m_q x}, x e^{m_q x}, x^2 e^{m_q x}, \dots, x^{k-1} e^{m_q x}$$

are the solutions of the DE.

(3) If both $\alpha + j\beta$ and $\alpha - j\beta$ are the roots of the auxiliary function, then

$$e^{\alpha x} \cos(\beta x), e^{\alpha x} \sin(\beta x)$$

are the solutions of the DE.

(4) If the multiplicities of $\alpha + j\beta$ is k and the multiplicities of $\alpha - j\beta$ is also k , then

$$\begin{aligned} &e^{\alpha x} \cos(\beta x), xe^{\alpha x} \cos(\beta x), x^2 e^{\alpha x} \cos(\beta x), \dots, x^{k-1} e^{\alpha x} \cos(\beta x) \\ &e^{\alpha x} \sin(\beta x), xe^{\alpha x} \sin(\beta x), x^2 e^{\alpha x} \sin(\beta x), \dots, x^{k-1} e^{\alpha x} \sin(\beta x) \end{aligned}$$

are the solutions of the DE.

Note: If $\alpha + j\beta$ is a root of a **real coefficient** polynomial,
then $\alpha - j\beta$ is also a root of the polynomial.

$$a_n(\alpha + j\beta)^n + a_{n-1}(\alpha + j\beta)^{n-1} + \cdots + a_1(\alpha + j\beta) + a_0 = 0$$

$a_0, a_1, a_2, \dots, a_n$ are real

Example 3 (text page 136)

Solve $y''' + 3y'' - 4y = 0$



Step 1-1 $m^3 + 3m^2 - 4 = 0$

$$(m-1)(m^2 + 4m + 4) = 0$$

$$m_1 = 1, \quad m_2 = m_3 = -2$$

Step 1-2 3 independent solutions: e^x, e^{-2x}, xe^{-2x}

Step 1-3 general solution: $y = c_1e^x + c_2e^{-2x} + c_3xe^{-2x}$

Example 4 (text page 137)

Solve

$$y^{(4)}(x) + 2y''(x) + y(x) = 0$$



Step 1-1

$$m^4 + 2m^2 + 1 = 0$$

$$(m^2 + 1)^2 = 0$$

four roots: $i, i, -i, -i$

Step 1-2

4 independent solutions: $\cos x, x \cos x, \sin x, x \sin x$

Step 1-3

general solution: $y = c_1 \cos x + c_2 x \cos x + c_3 \sin x + c_4 x \sin x$

4-3-5 How to Find the Roots

(1) Formulas

$$\boxed{a_2 m^2 + a_1 m + a_0 = 0} \Rightarrow m_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2 a_0}}{2a_2} \quad m_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}$$

$$\boxed{a_3 m^3 + a_2 m^2 + a_1 m + a_0}$$

$$\Rightarrow m_1 = S + T - \frac{a_2}{3a_3}$$

$$m_2 = -\frac{1}{2}(S + T) - \frac{a_2}{3a_3} + i\frac{1}{2}\sqrt{3}(S - T)$$

Solutions:

$$m_3 = -\frac{1}{2}(S + T) - \frac{a_2}{3a_3} - i\frac{1}{2}\sqrt{3}(S - T)$$

$$S = \sqrt[3]{R + \sqrt{Q^3 + R^3}} \quad , \quad T = \sqrt[3]{R - \sqrt{Q^3 + R^3}}$$

$$Q = \frac{1a_1}{3a_3} - \frac{a_2^2}{9a_3^2} \quad , \quad R = \frac{9a_1a_2 - 27a_3a_0 - 2a_2^3}{54a_3^2}$$

(2) Observing

例如：1 是否為 root $\longrightarrow$ 看係數和是否為 0

又如： $3m^3 + 5m^2 + 10m - 4$

factor: 1,3

factor: 1,2,4

possible roots: $\pm 1, \pm 2, \pm 4, \pm 1/3, \pm 2/3, \pm 4/3$

test for each possible root $\longrightarrow$ find that $1/3$ is indeed a root

$$3m^3 + 5m^2 + 10m - 4 = \left(m - \frac{1}{3}\right)(3m^2 + 6m + 12)$$

(3) Solving the roots of a polynomial by software

Maple

Mathematica (by the commands of **Nsolve** and **FindRoot**)

Matlab (by the command of **roots**)

4-3-6 本節需注意的地方

- (1) 注意重根和 conjugate complex roots 的情形
- (2) 寫解答時，要將 “General solution” 寫出來

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

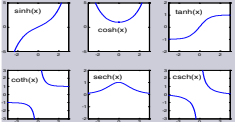
- (3) 因式分解要熟練
- (4) 本節的方法，也適用於 1st order 的情形

Exercise for practice

Sec. 4-1: 3, 7, 8, 13, 20, 24, 29, 33, 36

Sec. 4-2: 2, 4, 9, 13, 14, 18, 19

Sec. 4-3: 7, 16, 20, 24, 28, 33, 39, 41, 50, 51

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