Electromagnetics II

Unit 7：Potential Functions and Hertzian Dipole

Lecturer: Professor Jean-Fu Kiang

Text book: N. N. Rao, “Elements of Engineering Electromagnetics,” sixth ed., Pearson, 2004.

Wave Equation for Potential Function

In a homogeneous medium with μ and ε

$$\bar{B} = \mu \bar{H}$$

$$\bar{D} = \varepsilon \bar{E}$$

$$\nabla \times \bar{E} = -j\omega \bar{B} \rightarrow \nabla \times \bar{E} = -j\omega \nabla \times \bar{A}$$

$$\rightarrow \nabla \times (E + j\omega \bar{A}) = 0 \rightarrow E = -j\omega \bar{A} - \nabla \Phi \text{ (step 2)}$$

$$\nabla \times \bar{H} = j\omega \bar{D} + \bar{J} \rightarrow \nabla \times \nabla \times \bar{A}$$

$$= j\omega\mu\varepsilon(-\nabla\Phi - j\omega\bar{A}) + \mu\bar{J} \text{ (step 3)}$$

$$\nabla \cdot \bar{D} = \rho$$

$$\nabla \cdot \bar{B} = 0 \rightarrow \bar{B} = \nabla \times \bar{A} \text{ (step 1)}$$

$$\rightarrow \nabla\nabla \cdot \bar{A} - \nabla^2 \bar{A} = k^2 \bar{A} - j\omega\varepsilon\nabla\Phi + \mu\bar{J}$$

$$\rightarrow (\nabla^2 + k^2)\bar{A} = -\mu\bar{J}$$

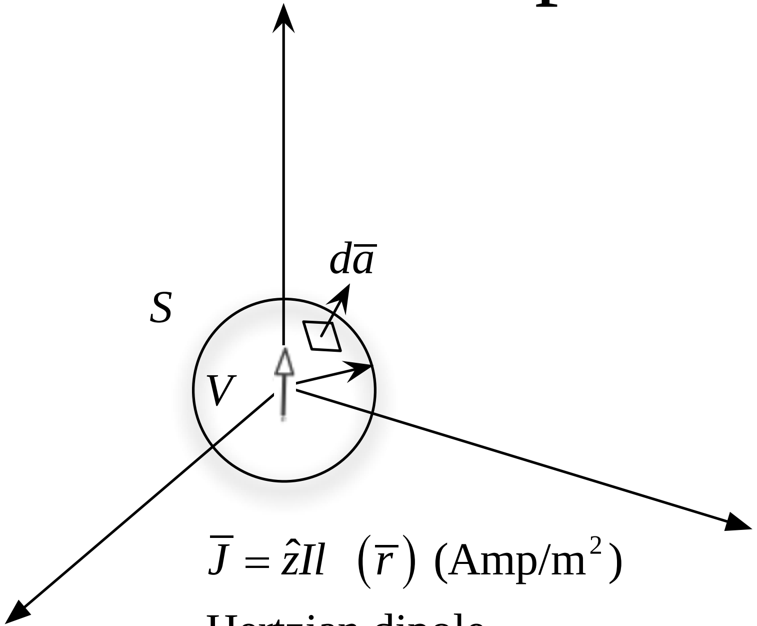
where $k^2 = \omega^2 \mu\varepsilon$

$$\nabla \times \nabla \times \bar{A} = \nabla\nabla \cdot \bar{A} - \nabla^2 \bar{A}$$

let $\nabla \cdot \bar{A} = -j\omega\varepsilon\Phi$ Lorentz gauge

also $\nabla \times \bar{A} = \bar{B} \rightarrow \bar{A}$ is unique

Hertzian Dipole



$\bar{J} = \hat{z}Il \ \ (\bar{r})$ (Amp/m^2)

Hertzian dipole

$I \to \infty, l \to 0, Il = \text{constant}$

Solution of Vector Potential

$$\bar{J}(\bar{r}) = \hat{z}Il\delta(\bar{r}) \rightarrow \bar{A}(\bar{r}) = \hat{z}A(\bar{r})$$

$$(\nabla^2 + k^2)A(\bar{r}) = -\mu Il\delta(\bar{r})$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2}{\partial\phi^2}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A}{\partial r}\right) + k^2A = -\mu Il\delta(\bar{r})$$

$$\text{if } \bar{r} \neq 0 \rightarrow \frac{d^2}{dr^2}(rA) + k^2(rA) = 0 \rightarrow rA = c_1e^{-jkr} + c_2e^{jkr}$$

$$A = c_1\frac{e^{-jkr}}{r}$$

$$\text{Re}\left\{rAe^{j\omega t}\right\} = c_1\cos(\omega t - kr) + c_2\cos(\omega t + kr) \rightarrow c_2 = 0$$

Spherically symmetric

$$\iiint_V (V^2 + k^2)c_1\frac{e^{-jkr}}{r}dr = \iiint_V -\mu Il\delta(\bar{r})d\bar{r}$$

$$\rightarrow c_1 = \frac{\mu Il}{4\pi} \rightarrow \bar{A} = \hat{z}\frac{\mu Il}{4\pi r}e^{-jkr}$$

$$\nabla^2 A = \nabla\cdot\nabla A$$

$$\iiint_V \nabla^2 A d\bar{r} = \oiint_S \nabla A\cdot d\bar{a}$$

$$= -c_1 4\pi e^{-jkr_0} \text{ with } r_0 \rightarrow 0$$

Expression of Magnetic Field

$$\overline{A} = \hat{z}\frac{\mu Il}{4\pi r}e^{-jkr}$$

$$\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$$

$$\overline{A} = (\hat{r}\cos\theta - \hat{\theta}\sin\theta)\frac{\mu Il}{4\pi r}e^{-jkr}$$

$$\overline{H} = \frac{1}{\mu}\nabla\times A = \frac{Il}{4\pi}\frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ \frac{e^{-jkr}}{r}\cos\theta & -e^{jkr}\sin\theta & 0 \end{vmatrix}$$

$$= \hat{\phi}\frac{jkIl}{4\pi r}e^{-jkr}\left(1+\frac{1}{jkr}\right)\sin\theta$$

Expression of Electric Field

$$\bar{E} = \frac{1}{j\omega\varepsilon}\nabla\times\bar{H} = \frac{1}{j\omega\varepsilon}\frac{jkIl}{4\pi}\frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ 0 & 0 & e^{-jkr}\left(1+\frac{1}{jkr}\right)\sin^2\theta \end{vmatrix} = \hat{r}E_r + \hat{\theta}E_\theta$$

$$E_r = \frac{1}{j\omega\varepsilon}\frac{jkIl}{4\pi}\frac{1}{r^2\sin\theta}e^{-jkr}\left(1+\frac{1}{jkr}\right)2\sin\theta\cos\theta = \eta\frac{jkIl}{4\pi}\frac{e^{-jkr}}{r}\left[\frac{1}{jkr}+\left(\frac{1}{jkr}\right)^2\right]2\cos\theta$$

$$E_\theta = \frac{-1}{j\omega\varepsilon}\frac{jkIl}{4\pi}\frac{1}{r\sin\theta}\sin^2\theta\frac{\partial}{\partial r}\left[e^{-jkr}\left(1+\frac{1}{jkr}\right)\right] = \eta\frac{jkIl}{4\pi r}e^{-jkr}\left[1+\frac{1}{jkr}+\left(\frac{1}{jkr}\right)^2\right]\sin\theta$$

Approximation in the Far-field Region

In the far-field region $kr \gg 1 \Rightarrow \dfrac{1}{jkr} \rightarrow 0$

$$E_r = 0$$

$$E_\theta = \eta \frac{jkIl}{4\pi r} e^{-jkr} \sin\theta = \eta H_\phi$$

$$\bar{E} \times \bar{H}^* = \hat{r}\eta \left(\frac{kIl}{4\pi r} \right)^2 \sin^2\theta$$

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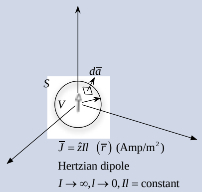
3		BY NC SA	**Jean-Fu Kiang / National Taiwan University** This work is licensed by Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Taiwan