Exercises for Unit 3

1. Let μ be additive on an algebra $\mathcal{A}$ on Ω.
   (a) Show that μ is σ-additive if and only if μ is continuous from below on $\mathcal{A}$.
   (b) Show that if μ is σ-additive on $\mathcal{A}$, then for any seq $\{A_n\} \subset \mathcal{A}$ with $\bigcup\limits_n A_n \in \mathcal{A}$ we have $\mu(\bigcup\limits_n A_n) \leq \sum\limits_n \mu(A_n)$ (sub σ-additivity).

2. Let μ be a σ-additivity set function on an algebra $\mathcal{A}$ on Ω with $\mu(\Omega) < \infty$. Suppose that μ_1 and μ_2 are measures on a σ-algebra $\Sigma \supset \mathcal{A}$ such that $\mu_1(A) = \mu_2(A) = \mu(A)$ for all $A \in \mathcal{A}$. Show that $\mu_1 = \mu_2$ on $\sigma(\mathcal{A})$. (Hint: Show that $\{B \in \Sigma : \mu_1(B) = \mu_2(B)\}$ is a λ-system on Ω)

3. Let (Ω, Σ, μ) be a measure space, and suppose that f and g are measurable functions with $f \leq g$.
   (a) If $\int_\Omega g^+ d\mu < \infty$, show that $\int_\Omega f d\mu$ exists and $\int_\Omega f d\mu \leq \int_\Omega g d\mu$.
   (b) Suppose that both $\int_\Omega f d\mu$ and $\int_\Omega g d\mu$ exist. Show that $\int_\Omega f d\mu \leq \int_\Omega f d\mu$

4. Let f be a measurable function on (Ω, Σ, μ). Show that $\int_\Omega f d\mu$ exists if and only if $f = f_1 - f_2$ for some nonnegative measurable functions f_1 and f_2 such that $\int_\Omega f_1 d\mu - \int_\Omega f_2 d\mu$ is meaningful. (Hint: for f_1 and f_2 as above, observe that $f^+ \leq f_1$ and $f^- \leq f_2$).

5. If f and g are measurable functions on (Ω, Σ, μ) such that $\int_\Omega f d\mu$, $\int_\Omega g d\mu$, and $\int_\Omega f d\mu + \int_\Omega g d\mu$ are meaningful, show that $f + g$ is defined a.e. and
$$\int_\Omega (f+g) d\mu = \int_\Omega f d\mu + \int_\Omega g d\mu$$
.

6. Let f be a measurable function on (Ω, Σ, μ).
   (a) If $f \geq 0$ a.e., then show that $\int_\Omega f d\mu = 0$ if and only if $f = 0$ a.e.
   (b) If $A \in \Sigma$, define $\int_A f d\mu = \int_\Omega f I_A d\mu$ if $\int_\Omega f I_A d\mu$ exits.
   Show that $f = 0$ a.e. if and only if $\int_A f d\mu = 0$ for all $A \in \Sigma$.

7. Suppose that f and g are defined a.e. on (Ω, Σ, μ) and are measurable. Show that if $f + g$ is defined a.e. then $f + g$ is measurable.

8. Show that if $\{f_n\}$ is a seq. of measurable functions which is bounded from below by an integrable function a.e. and is nondecreasing a.e., then $\int_\Omega \lim\limits_{n\to\infty} f_n d\mu = \lim\limits_{n\to\infty} \int_\Omega f_n d\mu$. (Hint: Show first that f_n^- is integrable and hence $\int_\Omega f_n d\mu$ is defined for each n.)

9. Let (Ω, Σ, μ) be a measure space, and let $\{A_k\}_{k=1}^\infty \subset \Sigma$.

(a) Show that if f is integrable, then

$$\int_{\limsup\limits_{k\to\infty} A_k} f d\mu = \lim_{k\to\infty} \int_{\bigcup\limits_{j=k}^{\infty} A_j} f d\mu.$$

(b) Let f be integrable and $\epsilon > 0$. Show that there is $\delta > 0$ such that if $A \in \Sigma$ and $\mu(A) < \delta$, then $\int_A |f| d\mu < \epsilon$.
(Hint: suppose the contrary. Then for each $k \in \mathbb{N}$, there is $A_k \in \Sigma$ such that $\mu(A_k) < \frac{1}{k^2}$ and $\int_A |f| d\mu \geq \epsilon$, then observe that $\mu(\limsup\limits_{k\to\infty} A_k) = 0$ and conclude a contradiction.)

10. (a) Show that for $1 \leq p < \infty$, $|a_1 + \cdots + a_n|^p \leq n^{p-1} \sum\limits_{j=1}^{n} |a_j|^p$ for real numbers $a_1, \cdots, a_n$.

    (b) Show that if $f, g \in L^p(\Omega, \Sigma, \mu)$, $1 \leq p < \infty$, with $\|f\|_p + \|g\|_p < \infty$, then $\|f + g\|_p < \infty$.

11. Let (Ω, Σ, μ) be a measure space with $\mu(\Omega) < \infty$ and suppose that $\{f_n\}$ is a seq. of measurable functions each of which takes finite value a.e. and that $f_n \to f$ a.e. with finite limit. Show that there are $Z, A_1, A_2, \cdots$ in Σ such that $\Omega = Z \bigcup \bigcup\limits_n A_n$, $\mu(Z) = 0$, and $f_n \to f$ uniformly on each A_k.

12. Suppose that $\{f_n\}$ is a seq. of measurable functions on (Ω, Σ, μ) . Show that if $\int_\Omega \sum\limits_{n=1}^{\infty} |f_n| d\mu < \infty$, then $\sum\limits_{n=1}^{\infty} f_n$ converges a.e., the function $\sum\limits_{n=1}^{\infty} f_n$ is integrable, and

$$\int_\Omega \sum_{n=1}^{\infty} f_n d\mu = \sum_{n=1}^{\infty} \int_\Omega f_n d\mu$$

13. Suppose that $\{f_k\}$ is a seq. in $L^p(\Omega, \Sigma, \mu)$, $1 \leq p < \infty$, such that $f_k \to f$ a.e. with $f \in L^p(\Omega, \Sigma, \mu)$ and $\|f\|_p = \lim\limits_{k\to\infty} \|f_k\|_p$. Show that $f_k \to f$ in $L^p(\Omega, \Sigma, \mu)$.

14. Suppose that the measure space (Ω, Σ, μ) is finite, i.e. $\mu(\Omega) < \infty$, and $f \in L^\infty(\Omega, \Sigma, \mu)$.

    (a) Show that $(\frac{1}{\mu(\Omega)} \int_\Omega |f|^p d\mu)^{\frac{1}{p}} \leq (\frac{1}{\mu(\Omega)} \int_\Omega |f|^{p'} d\mu)^{\frac{1}{p'}}$ if $1 \leq p \leq p' < \infty$.

    (b) Show that $\lim\limits_{p\to\infty} (\frac{1}{\mu(\Omega)} \int_\Omega |f|^p d\mu) = \|f\|_\infty$

15. Suppose that $1 \leq p < r$. Show that for any q strictly between p and r we have

$$L^q(\Omega, \Sigma, \mu) \subset L^p(\Omega, \Sigma, \mu) + L^r(\Omega, \Sigma, \mu).$$