Section 7.4 Matrix Representations of Linear Operators

Definition.
Let $V$ be a finite-dimensional vector space and $\mathcal{B}$ be a basis for $V$. For any vector $v$ in $V$, the vector $\Phi_B(v)$ is called the coordinate vector of $v$ relative to $\mathcal{B}$ and is denoted as $[v]_B$.

$\Phi_B : V \rightarrow \mathbb{R}^n$ defined as

$$\Phi_B(c_1v_1 + c_2v_2 + \cdots + c_nv_n) = [c_1 \ c_2 \ \cdots \ c_n]^T.$$

Property:
$[u + v]_B = [u]_B + [v]_B$ and $[cu]_B = c[u]_B$ for all $u, v \in V$ and scalar $c$. 
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Let $V$ be a finite-dimensional vector space and $\mathcal{B}$ be a basis for $V$. For any vector $v$ in $V$, the vector $\Phi_{\mathcal{B}}(v)$ is called the coordinate vector of $v$ relative to $\mathcal{B}$ and is denoted as $[v]_{\mathcal{B}}$.

**Property:**
$[u + v]_{\mathcal{B}} = [u]_{\mathcal{B}} + [v]_{\mathcal{B}}$ and $[cu]_{\mathcal{B}} = c[u]_{\mathcal{B}}$ for all $u, v \in V$ and scalar $c$.

**Example:** Let $V = \text{Span} \ \mathcal{B}$, where $\mathcal{B} = \{e^t \cos t, e^t \sin t\}$ is L.I. and thus a basis of $V$.
Consider the function $v = e^t \cos (t - \pi /4)$. (Is it in $V$?)
Then $v$ is in $V$ since

$$v = \frac{1}{\sqrt{2}} e^t (\cos t + \sin t) = \frac{1}{\sqrt{2}} e^t \cos t + \frac{1}{\sqrt{2}} e^t \sin t$$

In addition, $[v]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T$
Consider a linear transformation $T: V \rightarrow W$

Questions:
1) Can we define a “standard matrix” for $T$?
2) If not, what kind of matrix representation of $T$ can we formulate?

In this course, we will consider only a simpler case where $T$ is a linear operator (i.e., the domain and the codomain are the same vector space).
Let $T : V \rightarrow V$ be a linear operator on an $n$-dimensional vector space $V$ with a basis $\mathcal{B}$. Define the linear operator $\Phi_B T (\Phi_B)^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$, and consider its standard matrix $A$, called the matrix representation of $T$ with respect to $\mathcal{B}$ and denoted as $[T]_B$. With the notations, $[T]_B = A$ and $T_A = \Phi_B T (\Phi_B)^{-1}$. 

![Diagram showing the relationship between $T$, $\Phi_B$, $(\Phi_B)^{-1}$, and $[T]_B$.](image)
Let $T : V \rightarrow V$ be a linear operator on an $n$-dimensional vector space $V$ with a basis $\mathcal{B}$. Define the linear operator $\Phi_{\mathcal{B}} T (\Phi_{\mathcal{B}})^{-1} : \mathcal{R}^n \rightarrow \mathcal{R}^n$, and consider its standard matrix $A$, called the matrix representation of $T$ with respect to $\mathcal{B}$ and denoted as $[T]_{\mathcal{B}}$. With the notations, $[T]_{\mathcal{B}} = A$ and $T_A = \Phi_{\mathcal{B}} T (\Phi_{\mathcal{B}})^{-1}$.

Question: How to express $[T]_{\mathcal{B}}$ in terms of $T$ and $b_1, b_2, \ldots, b_n$?
Property:
If $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$, then $[T]_\mathcal{B} = [ [T(v_1)]_\mathcal{B} \ [T(v_2)]_\mathcal{B} \ \cdots \ [T(v_n)]_\mathcal{B} ]$.

**Proof** $[T]_\mathcal{B} = A \Rightarrow A e_j = T_A(e_j) = \Phi_\mathcal{B} T(\Phi_\mathcal{B})^{-1}(e_j) = \Phi_\mathcal{B} T(v_j) = [T(v_j)]_\mathcal{B}$. 

Example: Let $T: \mathcal{P}_2 \to \mathcal{P}_2$ be defined by $T(p(x)) = p(0) + 3p(1)x + p(2)x^2$ for all $p(x)$ in $\mathcal{P}_2$. Then $T$ is linear. For $\mathcal{B} = \{1, x, x^2\}$, 

$[T]_\mathcal{B} = A = [ \begin{array}{ccc} a_1 & a_2 & a_3 \end{array}]$ and

$a_1 = [T(1)]_\mathcal{B} = [1 + 3x + x^2]_\mathcal{B} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$, $a_2 = [T(x)]_\mathcal{B} = [3x + 2x^2]_\mathcal{B} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, 

$a_3 = [T(x^2)]_\mathcal{B} = [3x + 4x^2]_\mathcal{B} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$, so $[T]_\mathcal{B} = [a_1 \ a_2 \ a_3] = \begin{bmatrix} 0 \\ 3 \\ 3 \\ 3 \end{bmatrix}$. 

$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 4 \end{bmatrix}$
Example: Let $V = \text{Span } \mathcal{B}$, where $\mathcal{B} = \{e^t \cos t, e^t \sin t\}$ is L.I. and thus a basis of $V$, and the linear operator $D: V \to V$ be defined by $D(f) = f''$ for all $f \in V$. Then
\begin{align*}
D(e^t \cos t) &= (1)e^t \cos t + (-1)e^t \sin t \\
D(e^t \sin t) &= (1)e^t \cos t + (1)e^t \sin t
\end{align*}
$\Rightarrow [D]_B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$
Theorem 7.10
Let $T$ be a linear operator on a finite-dimensional vector space $V$ with basis $\mathcal{B}$. Then for any vector $v$ in $V$,

$$[T(v)]_B = [T]_B[v]_B.$$ 

Proof  $[T(v)]_B = \Phi_B T(v) = \Phi_B T(\Phi_B)^{-1} \Phi_B(v) = T_A([v]_B) = [T]_B[v]_B,$

where $A = [T]_B$. 
Example: Relative to the basis $\mathcal{B} = \{1, x, x^2\}$ of $\mathcal{P}_2$, the coordinate vector of $p(x) = 5 - 4x + 3x^2$ is $[p(x)]_{\mathcal{B}} = [5 \ -4 \ 3]^T$.

Then $[p'(x)]_{\mathcal{B}} = [D(p(x))]_{\mathcal{B}} = [D]_B [p(x)]_{\mathcal{B}}$, where $D: \mathcal{P}_2 \to \mathcal{P}_2$ is defined by $D(p(x)) = p'(x)$, and

$$[D]_B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow [p'(x)]_B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \\ 0 \end{bmatrix}.$$
The Matrix Representation of the Inverse of an Invertible Linear Operator

Let $T$ be a linear operator on a finite-dimensional vector space $V$ with basis $\mathcal{B}$ and let $A = [T]_\mathcal{B}$. Then the following statements are true.

(a) $T$ is invertible if and only if $A$ is invertible.
(b) If $T$ is invertible, then $[T^{-1}]_\mathcal{B} = A^{-1}$.

Proof (a) Note that $\Phi_\mathcal{B}$ is an isomorphism with an inverse $(\Phi_\mathcal{B})^{-1}$, which is also an isomorphism.

If $T$ is invertible, then $T_A = \Phi_\mathcal{B} T (\Phi_\mathcal{B})^{-1}$ is a composition of isomorphisms. So $T_A$ is invertible and has an invertible standard matrix $A$.

If $A$ is invertible, then $T_A = \Phi_\mathcal{B} T (\Phi_\mathcal{B})^{-1}$ is invertible. So $T = (\Phi_\mathcal{B})^{-1} T_A \Phi_\mathcal{B}$ is invertible.
The Matrix Representation of the Inverse of an Invertible Linear Operator

Let $T$ be a linear operator on a finite-dimensional vector space $V$ with basis $\mathcal{B}$ and let $A = [T]_\mathcal{B}$. Then the following statements are true.

(a) $T$ is invertible if and only if $A$ is invertible.
(b) If $T$ is invertible, then $[T^{-1}]_\mathcal{B} = A^{-1}$.

**Proof**  (b) By (a) and the invertibility of $T$, $T_C = \Phi_\mathcal{B} T^{-1} (\Phi_\mathcal{B})^{-1}$, where $C = [T^{-1}]_\mathcal{B}$.

Also by (a), $T^{-1} = (T_A)^{-1} = \Phi_\mathcal{B} T^{-1} (\Phi_\mathcal{B})^{-1}$.

$\Rightarrow T_C = T_A^{-1} \Rightarrow C = A^{-1}$.
Example: In the vector space $V$ with a basis $\mathcal{B} = \{e^t \cos t, e^t \sin t\}$ and a linear operator $D: V \to V$ defined by $D(f) = f' \quad \forall f \in V$,

it is known that $[D]_\mathcal{B} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. So an anti-derivative of $e^t \sin t$ is $D^{-1}(e^t \sin t)$.

Since $[D^{-1}]_\mathcal{B} = ([D]_\mathcal{B})^{-1}$ and $[e^t \sin t]_\mathcal{B} = [0 \ 1]^T$,

$$[D^{-1}(e^t \sin t)]_\mathcal{B} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$$

i.e., $D^{-1}(e^t \sin t) = -(e^t \cos t)/2 + (e^t \sin t)/2$. 
Definition.

For a linear operator $T$ on a vector space $V$ (over a field $\mathcal{F}$), a nonzero vector $v$ in $V$ is said to be an eigenvector of $T$ corresponding to the eigenvalue $\lambda$ if there is a scalar $\lambda \in \mathcal{F}$ such that $T(v) = \lambda v$. For an eigenvalue $\lambda$ of $T$, the set of all vectors $v \in V$ satisfying $T(v) = \lambda v$ is the eigenspace of $T$ corresponding to $\lambda$.

Example: For linear operator $D: C^\infty \to C^\infty$ defined by $D(f) = f'$ with $C^\infty = \{ f | f: \mathbb{R} \to \mathbb{R}, f \text{ has derivatives of all order} \}$, it has an eigenvector $e^{\lambda t} \in C^\infty$ corresponding to the eigenvalue $\lambda$, since $D(f(t)) = (e^{\lambda t})' = \lambda e^{\lambda t} = \lambda f(t)$.

$\Rightarrow$ Any scalar $\lambda$ is an eigenvalue of $D$.

$\Rightarrow$ $D$ has infinitely many eigenvalues.
Example: $f$ is a solution of $y'' + 4y = 0 \Rightarrow f \in C^\infty$, since $f$ must be twice-differentiable and $f''' = -4f$, which imply that the fourth derivative of $f$ exists ($f'''' = -4f'''$), and so on. 

$\Rightarrow f \neq 0$ is an eigenvector of $D^2 : C^\infty \rightarrow C^\infty$ corresponding to the eigenvalue -4.

$\Rightarrow f \in$ eigenspace of $D^2$ corresponding to the eigenvalue -4.

Clearly, every vector in the eigenspace of $D^2$ corresponding to the eigenvalue -4 is a solution of $y'' + 4y = 0$. 
Example: \( U: \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n} \) defined by \( U(A) = A^T \) is an isomorphism.

The eigenspace of \( U \) corresponding to the eigenvalue 1 is the set of \( A \in \mathbb{R}^{n \times n} \) such that \( U(A) = A^T = A \), i.e., the set of symmetric matrices.

The eigenspace of \( U \) corresponding to the eigenvalue -1 is the set of \( A \in \mathbb{R}^{n \times n} \) such that \( U(A) = A^T = -A \), i.e., the set of skew-symmetric matrices.

\( U \) only has eigenvalues 1 and -1, since \( U(A) = A^T = \lambda A \) implies \( A = (A^T)^T = (\lambda A)^T = \lambda A^T = \lambda (\lambda A) = \lambda^2 A. \)
Eigenvalues and Eigenvectors of a Matrix Representation of a Linear Operator

Let $T$ be a linear operator on a finite-dimensional vector space $V$ with basis $B$ and let $A = [T]_B$. Then a vector $v$ in $V$ is an eigenvector of $T$ with corresponding eigenvalue $\lambda$ if and only if $[v]_B$ is an eigenvector of $A$ with corresponding eigenvalue $\lambda$.

**Proof** “only if” ($\implies$) Suppose $v \neq 0$ satisfies $T(v) = \lambda v$.

$\implies [v]_B \neq 0$ and $A[v]_B = [T]_B[v]_B = [T(v)]_B = [\lambda v]_B = \lambda [v]_B$

“if” ($\Leftarrow$) Suppose $w \neq 0$ satisfies $Aw = [T]_B w = \lambda w$.

Let $v = (\Phi_B)^{-1}(w) \neq 0$.

$\implies \Phi_B T(v) = [T(v)]_B = [T]_B[v]_B = \lambda [v]_B = \Phi_B (\lambda v)$

$\implies T(v) = \lambda v$ since $\Phi_B$ is an isomorphism.
Example: Let \( T: P_2 \rightarrow P_2 \) be defined as \( T(p(x)) = p(0) + 3p(1)x + p(2)x^2 \) for all \( p(x) \) in \( P_2 \). Then \( T \) is linear, and

\[
[T]_B = A = \begin{bmatrix}
1 & 0 & 0 \\
3 & 3 & 3 \\
1 & 2 & 4
\end{bmatrix}
\]

where \( B = \{1, x, x^2\} \) is a basis of \( P_2 \).

The characteristic polynomial of \( A \) is \(- (t - 1)^2(t - 6)\).

\( \text{Span}\{[ 0 \ -3 \ 2 ]^T\} \) is the eigenspace of \( A \) corresponding to the eigenvalue 1 \( \Rightarrow ap(x) \) with \( a \neq 0 \) and \( p(x) = -3x + 2x^2 \) is an eigenvector of \( T \) corresponding to the eigenvalue 1.

\( \text{Span}\{[ 0 \ 1 \ 1 ]^T\} \) is the eigenspace of \( A \) corresponding to the eigenvalue 6 \( \Rightarrow bq(x) \) with \( b \neq 0 \) and \( q(x) = x + x^2 \) is an eigenvector of \( T \) corresponding to the eigenvalue 6.
Example: $U: \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^{2 \times 2}$ defined by $U(A) = A^T$ is an isomorphism and only has eigenvalues 1 and -1.

Let a basis of $\mathbb{R}^{2 \times 2}$ be

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\Rightarrow [U]_B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow$$ The characteristic polynomial of $[U]_B$ is $(t - 1)^3(t + 1)$, for which indeed the only roots are 1 and -1.
Homework Set for Section 7.4

- Section 7.4: Problems 1, 5, 9, 11, 19, 21, 23, 28, 32, 36, 39, 40, 43, 44, 46.