Section 4.3 The Dimension of Subspaces Associated with a Matrix

Subspaces associated with a matrix A: Col A, Null A, Row A.

The dimension of the column space of a matrix equals the rank of the matrix.

$$\dim\ (\text{Col}\ \ A) = \text{rank} A$$

Proof Pivot columns form a basis of the column space.

Example:

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix} \Rightarrow \text{Col } A = \text{Span}\left\{ \underbrace{\begin{bmatrix} 1 \\ -1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}}_{\text{pivot columns}} \right\}$$

$\Rightarrow$ dim. Col $A = 3$

The dimension of the null space of a matrix equals the nullity of the matrix.

Proof Nullity of A is the number of free variables in $A\mathbf{x} = \mathbf{0}$, and each free variable in the parametric form of the general solution is multiplied by a vector in a basis for the solution set.

Example: $\mathcal{B}$ is a basis of V, where

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\} \quad V = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \in \mathcal{R}^4 : v_1 + v_2 + v_4 = 0 \right\}$$

In fact, $A = [\ 1 \ \ 1 \ \ 0 \ \ 1\ ]$ and rank $A = 1 \Rightarrow$ nullity $A = 3 \Rightarrow$ dim. $V = 3$

Example: Is $\mathcal{B}$ is a basis of Null A?

$$\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ -2 \\ -2 \\ -1 \end{bmatrix} \right\} \quad A = \begin{bmatrix} 3 & 1 & -2 & 1 & 5 \\ 1 & 0 & 1 & 0 & 1 \\ -5 & -2 & 5 & -5 & -3 \\ -2 & -1 & 3 & 2 & -10 \end{bmatrix}$$

1. $\mathcal{B} \subseteq$ Null A ($\mathbf{x} \in \mathcal{B} \Rightarrow A\mathbf{x} = \mathbf{0}$.)
2. $\mathcal{B}$ is L.I., as neither vector in $\mathcal{B}$ is a multiple of each other.
3. Nullity of $A = 2$ (you check that rank $A = 3$.)

$\Rightarrow \mathcal{B}$ is a basis of Null A.

Row A: the subspace spanned by the rows of A.

Property: Row A = Row EA for any elementary matrix E.

Proof Rows of EA are linear combinations of rows of A.

$\Rightarrow$ Row $EA \subseteq$ Row A by Theorem 1.6(b) (Section 1.6).

Also, $A = E^{-1}EA$ and E^{-1} is an elementary matrix.

$\Rightarrow$ Row $A \subseteq$ Row EA.

Property: In general Col $A \neq$ Col EA.

Example:

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \xrightarrow[\text{operation}]{\text{one elementary}} R = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\text{Row } A = \text{Row } R = \text{Span } \{[\ 1 \quad 1\ ]\}$$

$$\text{Col } A = \text{Span } \begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq \text{Col } R = \text{Span } \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

Preview Question

Consider an $m \times n$ matrix A and its reduced row echelon form R. We have learned that a basis of Col A can be found by collecting all of its pivot columns. How can we find a basis for Row A?

1) Since Row $A =$ Row R, a basis for Row A can be found by selecting all rows of R that contain a pivot entry.
2) Since Row $A =$ Row R, a basis for Row A can be found by selecting all nonzero rows of R.
3)All of the above.
4) None of the above.

Theorem 4.8

The nonzero rows of the reduced row echelon form of a matrix form a basis for the row space of the matrix.

Proof Let the reduced row echelon form of $A \in \mathcal{R}^{m\times n}$ be R, which is obtained from A by elementary operations $\Rightarrow$ Row R = Row A. Also, Row R = Span{nonzero rows of R}, and nonzero rows of R are L.I. (no nonzero row of R is a linear combination of other rows).

Corollary:

The dimension of the row space of a matrix equals its rank.

Example:

$$A = \begin{bmatrix} 3 & 1 & -2 & 1 & 5 \\ 1 & 0 & 1 & 0 & 1 \\ -5 & -2 & 5 & -5 & -3 \\ -2 & -1 & 3 & 2 & -10 \end{bmatrix} \xrightarrow{\text{reduced row echelon form}} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -5 & 0 & 4 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\{\cdots\}$: a basis of Row A

The rank of any matrix equals the rank of its transpose.

Proof rank A = dim. (Row A) = dim. (Col A) = dim. (Row A^T)
= rank A^T

The results in this Section may be extended to linear transformations $T: \mathcal{R}^n \to \mathcal{R}^m$ by considering their standard matrices.

Preview Question

Let V and W are both subspaces of $\mathcal{R}^n$ and $V \subset W$.

Q: What is the relationship between dim V and dim W?

Answer

1. dim $V \leq$ dim W.
2. dim $V <$ dim W.
3. We can't say anything about this.
4. None of the above.

Theorem 4.9

If V and W are subspaces of $\mathcal{R}^n$ such that V is contained in W, then dim $V \leq$ dim W. More over, if V and W also have the same dimension, then $V = W$.

Proof If $V = \{\mathbf{0}\}$ then the Theorem holds. Suppose $V \neq \{\mathbf{0}\}$.

Let $\mathcal{B}$ be a basis of V. By the Extension Theorem, $\mathcal{B} \subseteq$ a basis of W.

$\Rightarrow$ dim $V \leq$ dim W

Suppose dim. $V =$ dim $W = k$. $\Rightarrow$ $\mathcal{B}$ is L.I. and has k vectors in W.

$\Rightarrow$ $\mathcal{B}$ is a basis of W. $\Rightarrow$ $W = \text{Span}\,\mathcal{B} = V$.

Homework Set for Section 4.3

Section 4.3: Problems 1, 4, 6, 9, 15, 61, 64, 73, 81