1.6 Span of a Set of Vectors

Definition

For a nonempty set $\mathcal{S} = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ of vectors in $\mathcal{R}^n$, we define the **span of $\mathcal{S}$** to be the set of all linear combinations of $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k$ in $\mathcal{R}^n$. This set is denoted by Span $\mathcal{S}$ or Span $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$.

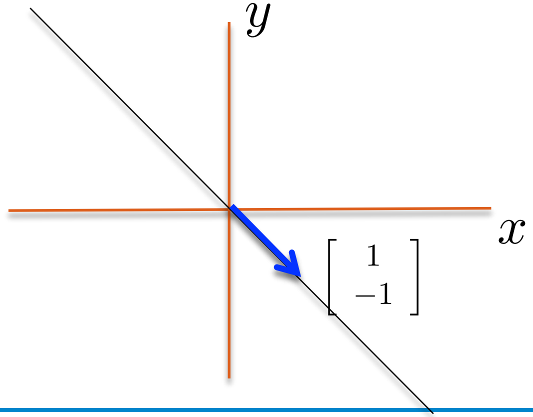
- Why did we (mathematicians) define the term "Span"?
  - When we want to say
    "$\mathbf{v}$ is a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$,"
    we can simply say

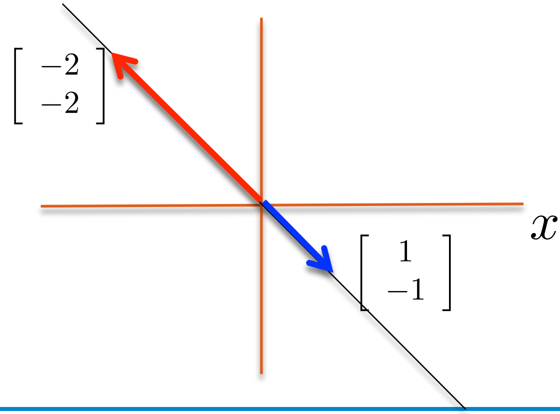
$$\mathbf{v} \in \text{Span } \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$$

$$\text{Span } \mathcal{S} = \{c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k : c_1, c_2, ..., c_k \in \mathcal{R}\}$$

Example

Let $\mathcal{S}_1 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$, what is Span $\mathcal{S}_1$?





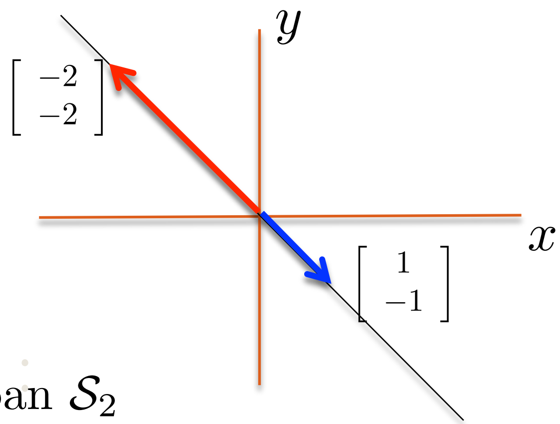
Example

How about $\mathcal{S}_2 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right\}$?

Example

How about $\mathcal{S}_3 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$?

Example:



$$\mathcal{S}_1 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}, \mathcal{S}_2 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right\}, \text{Span } \mathcal{S}_1 = \text{Span } \mathcal{S}_2$$

$$\mathcal{S}_3 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}, \mathcal{S}_4 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\},$$

$$\Rightarrow \text{Span} \mathcal{S}_3 = \text{Span} \mathcal{S}_4 = \mathcal{R}^2$$

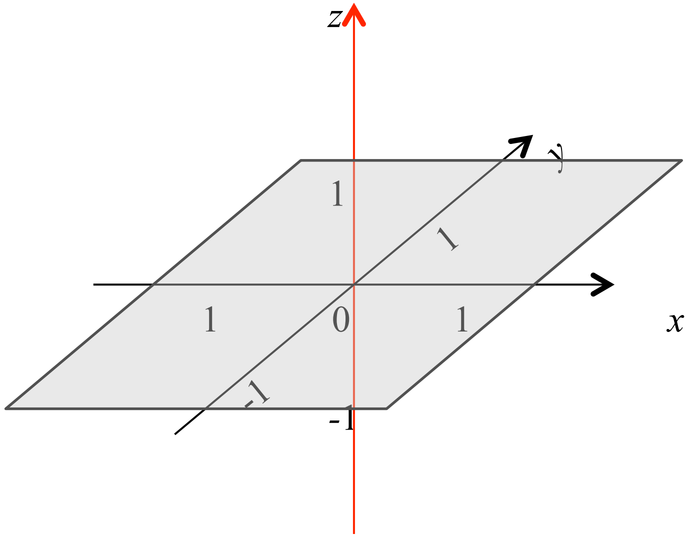
nonparallel vectors

Example: Consider the standard vectors in $\mathcal{R}^3$.

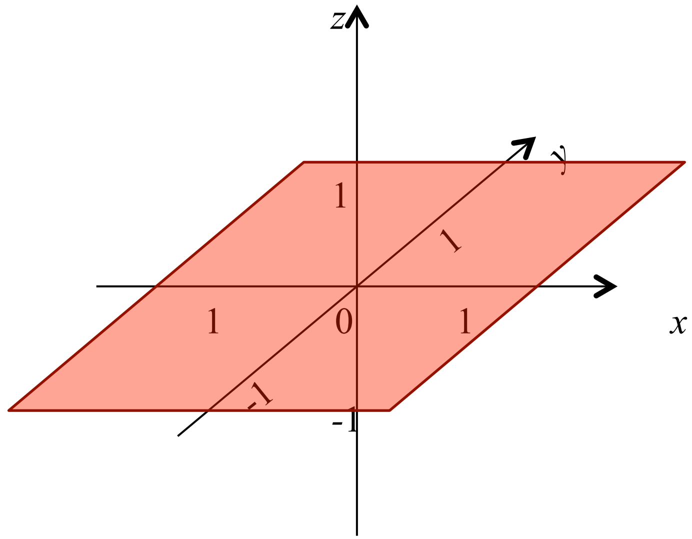
$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Span$\{\mathbf{e}_1, \mathbf{e}_2\}$ = xy-plane in $\mathcal{R}^3$
Span$\{\mathbf{e}_3\}$ = z-axis in $\mathcal{R}^3$

Span $\{\mathbf{e}_3\}$

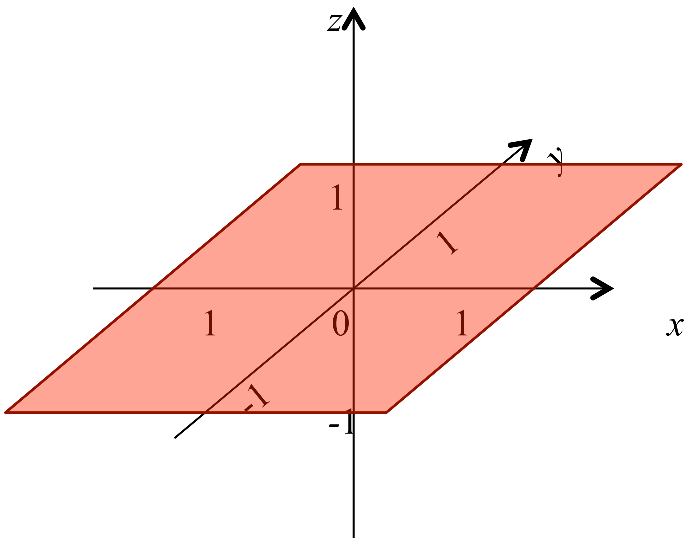


Span $\{\mathbf{e}_1, \mathbf{e}_2\}$

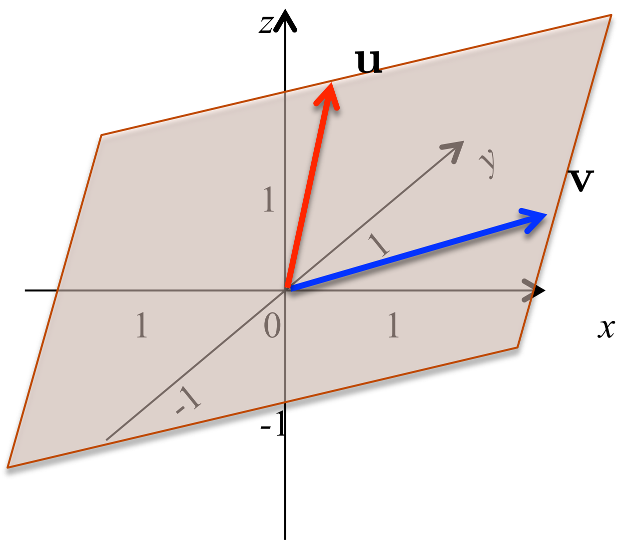


Examples: spans of vectors in R^3

Span $\{\mathbf{e}_1, \mathbf{e}_2\}$



Span $\{\mathbf{u}, \mathbf{v}\}$



Definition

For a nonempty set $\mathcal{S} = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ of vectors in $\mathcal{R}^n$, we define the **span of $\mathcal{S}$** to be the set of all linear combinations of $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k$ in $\mathcal{R}^n$. This set is denoted by Span $\mathcal{S}$ or Span $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$.

$$\text{Span } \mathcal{S} = \{c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k : c_1, c_2, \cdots, c_k \in \mathcal{R}\}$$

Properties

1. Span $\{\mathbf{0}\} = \{\mathbf{0}\}$
2. Span $\{\mathbf{u}\}$ = the set of all multiples of $\mathbf{u}$.
3. If $\mathcal{S}$ contains a nonzero vector, then Span $\mathcal{S}$ has infinitely many vectors.

Note that Span $\mathcal{S}$ can also be expressed as

$$\text{Span } \mathcal{S} = \left\{A\mathbf{v} : \mathbf{v} \in \mathcal{R}^k\right\}$$

where $A = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{bmatrix}$ is an $n \times k$ matrix.

Example: Let $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 8 \\ -1 \\ 5 \end{bmatrix} \right\}$, $\mathbf{v} = \begin{bmatrix} 3 \\ 0 \\ 5 \\ -1 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ -1 \end{bmatrix}$.

(1) Is $\mathbf{v} \in \text{Span } \mathcal{S}$? (2) Is $\mathbf{w} \in \text{Span } \mathcal{S}$?

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 8 \\ 1 & -2 & -1 \\ 1 & 1 & 5 \end{bmatrix}$$

Does $A\mathbf{x} = \mathbf{v}$ has a solution?

Is the system of linear equations $A\mathbf{x} = \mathbf{v}$ consistent?

Is the solution set of $A\mathbf{x} = \mathbf{v}$ non-empty?

The reduced row echelon form of $[\ A \ \ \mathbf{v}\ ]$ is

$$\begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \mathbf{v} \in \text{Span } \mathcal{S}$

The reduced row echelon form of $[\ A \ \ \mathbf{w}\ ]$ is

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \mathbf{w} \notin \text{Span } \mathcal{S}$

Definitions

If $\mathcal{S}, \mathcal{V} \subset \mathcal{R}^n$ and Span $\mathcal{S} = \mathcal{V}$, then we say
"$\mathcal{S}$ is a **generating set** for $\mathcal{V}$,"
or "$\mathcal{S}$ **generates** $\mathcal{V}$."

Example: Does $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$ generate $\mathcal{R}^3$?

Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

For any $\mathbf{v}$ in $\mathcal{R}^3$, let $[\ R\ \ \mathbf{c}\ ]$ be the reduced row echelon form of $[\ A\ \ \mathbf{v}\ ]$, then

$$R = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{ : rank} = 3$$

$\Rightarrow [\ R\ \ \mathbf{c}\ ]$ has no nonzero rows with only entries from $\mathbf{c}$

$\Rightarrow$ for any $\mathbf{v}$ in $\mathcal{R}^3$, $\mathbf{v} = A\mathbf{x}$ for some $\mathbf{x}$, thus Span $\mathcal{S} = \mathcal{R}^3$.

- Question
  - In general, on which conditions can a finite subset of $\mathcal{R}^m$, $\mathcal{S} = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\} \subset \mathcal{R}^m$, generate $\mathcal{R}^m$?
  - Let us define $A = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{bmatrix}$.
  - What is the span of columns of A?
  - When is $A\mathbf{x} = \mathbf{v}$ consistent for some $\mathbf{v} \in \mathcal{R}^m$?
  - When is $A\mathbf{x} = \mathbf{v}$ consistent for any $\mathbf{v} \in \mathcal{R}^m$?
  - What is the rank of A?
  - What property should the reduced row echelon form of A possess?

Theorem 1.6

The following statements about an $m \times n$ matrix A are equivalent.

(a) The **span** of the columns of A is $\mathcal{R}^m$.

(b) The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution (that is, $A\mathbf{x} = \mathbf{b}$ is consistent) for each $\mathbf{b} \in \mathcal{R}^m$.

(c) The **rank** of A is m, the number of rows of A.

(d) The **reduced row echelon form** of A has no zero rows.

(e) There is a **pivot position** in each row of A.

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The following statements about an $m \times n$ matrix A are equivalent.

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(b) The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution (that is, $A\mathbf{x} = \mathbf{b}$ is consistent) for each $\mathbf{b} \in \mathcal{R}^m$.

(c) The **rank** of A is m, the number of rows of A.

(d) The **reduced row echelon form** of A has no zero rows.

(e) There is a **pivot position** in each row of A.

Proof (a) $\Leftrightarrow$ (b): for any $\mathbf{b}$ in $\mathcal{R}^m$, $\mathbf{b} = A\mathbf{x}$ for some $\mathbf{x}$.

(c) $\Leftrightarrow$ (d): by the definition of rank.

(b) $\Rightarrow$ (c): otherwise, the reduced row echelon form R of A has a zero bottom row, and $R\mathbf{x} = \mathbf{e}_m$ is inconsistent $\Rightarrow$ with the inverse of the sequence of elementary oper`ations that transforms A to R, $[\,R \;\; \mathbf{e}_m\,]$ may be transformed back to $[\,A \;\; \mathbf{d}\,]$ with some $\mathbf{d}$ in $\mathcal{R}^m$, and $A\mathbf{x} = \mathbf{d}$ is inconsistent $\Rightarrow$ ⌧ (contradiction)

(c) $\Rightarrow$ (b): the reduced row echelon form R of A has no zero rows $\Rightarrow$ rank A = rank $[\,A \;\; \mathbf{b}\,]$ for all $\mathbf{b}$ in $\mathcal{R}^m$ $\Rightarrow$ (b).

(d) $\Leftrightarrow$ (e): by the definition of pivot position.

- Question
  - On which conditions does a finite subset of $\mathcal{R}^m$, $\mathcal{S}$, be the **smallest** set that generates $\mathcal{R}^m$?
  - Let us define $A = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{bmatrix}$.
  - On which conditions does a finite subset of $\mathcal{R}^m$, $\mathcal{S}$, be the **smallest** set that generates Span $\mathcal{S}$?
  - What is the feature of a "redundant" vector $\mathbf{v}$ in $\mathcal{S}$ when it comes to generating Span $\mathcal{S}$?
  - That is, what kind of $\mathbf{v} \in \mathcal{S}$ will make Span $\mathcal{S}$ = Span $(\mathcal{S} \setminus \{\mathbf{v}\})$?

Theorem 1.7

Let $\mathcal{S} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be a set of vectors from $\mathcal{R}^n$ and let $\mathbf{v}$ be a vector in $\mathcal{R}^n$. Then Span $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k, \mathbf{v}\}$ = Span $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ if and only if $\mathbf{v}$ belongs to the span of $\mathcal{S}$.

Proof

Theorem 1.7

Let $\mathcal{S} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be a set of vectors from $\mathcal{R}^n$ and let $\mathbf{v}$ be a vector in $\mathcal{R}^n$. Then Span $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k, \mathbf{v}\}$ = Span $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ if and only if $\mathbf{v}$ belongs to the span of $\mathcal{S}$.

Proof Since Span $\mathcal{S} \subseteq$ Span$(\mathcal{S} \cup \{\mathbf{v}\})$, only need to show

Span$(\mathcal{S} \cup \{\mathbf{v}\}) \subseteq$ Span $\mathcal{S} \Leftrightarrow \mathbf{v} \in$ Span $\mathcal{S}$.

"$\mathbf{v} \in$ Span$\mathcal{S}$":

"$\mathbf{v} \notin$ Span$\mathcal{S}$":

Theorem 1.7

Let $\mathcal{S} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be a set of vectors from $\mathcal{R}^n$ and let $\mathbf{v}$ be a vector in $\mathcal{R}^n$. Then Span $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k, \mathbf{v}\}$ = Span $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ if and only if $\mathbf{v}$ belongs to the span of $\mathcal{S}$.

Proof Since Span$\mathcal{S} \subseteq$ Span($\mathcal{S} \cup \{\mathbf{v}\}$), we only need to show Span($\mathcal{S} \cup \{\mathbf{v}\}$) $\subseteq$ Span $\mathcal{S} \Leftrightarrow \mathbf{v} \in$ Span $\mathcal{S}$.

"$\mathbf{v} \in$ Span$\mathcal{S}$":

Then $\mathbf{v} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_k\mathbf{u}_k$ for some scalars $a_1, a_2, \ldots, a_k$.

For all $\mathbf{w} \in$ Span($\mathcal{S} \cup \{\mathbf{v}\}$), $\mathbf{w} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k + b\mathbf{v}$ for some scalars $c_1, c_2, \ldots, c_k$ and b. Then $\mathbf{w} = (c_1 + ba_1)\mathbf{u}_1 + (c_2 + ba_2)\mathbf{u}_2 + \cdots + (c_k + ba_k)\mathbf{u}_k \in$ Span$\mathcal{S}$

"$\mathbf{v} \notin$ Span$\mathcal{S}$": Span($\mathcal{S} \cup \{\mathbf{v}\}$)} $\not\subseteq$ Span$\mathcal{S}$ since $\mathbf{v} \in$ Span($\mathcal{S} \cup \{\mathbf{v}\}$)

This also means by removing from a set some vectors which are linear combinations of other vectors, the span of the set is not changed.

Homework Set for 1.6

Section 1.6: Problems 1, 3, 9, 11, 17, 21, 25, 27, 31, 33, 35, 37, 39, 41, 45, 49, 53, 57, 61, 63, 69, 71