流體力學 Fluid Mechanics

Basic Concepts of Fluid Flow

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1. Basic concepts of fluid flow

1.1 Fluids vs solids

1.2 Continuum – number density
Local thermodynamic equilibrium
Pressure, temperature
Fields – density, pressure, temperature, velocity

1.3 Streamlines, pathlines, streaklines, material lines

1.4 Fluid motion: stress and strain rate

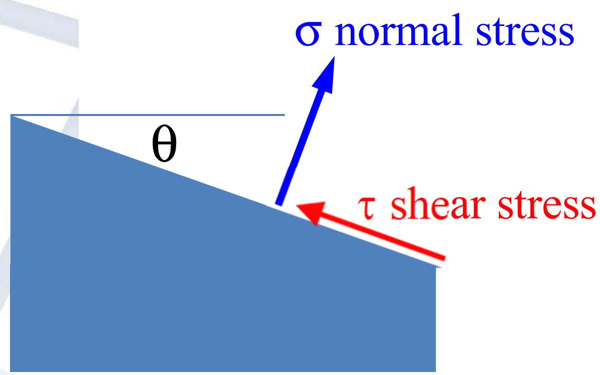
1.5 Dimensional Analysis: Buckingham Pi Theorem

1.6 Dimensionless parameters

§1.1 fluid vs solid

A **solid** can resist a shear stress by a **static** deformation.

Take an element A：

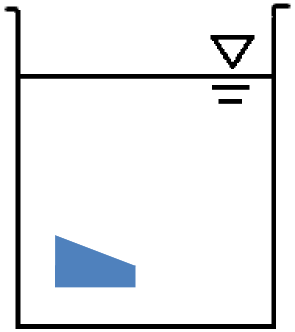


stress
vs
strain

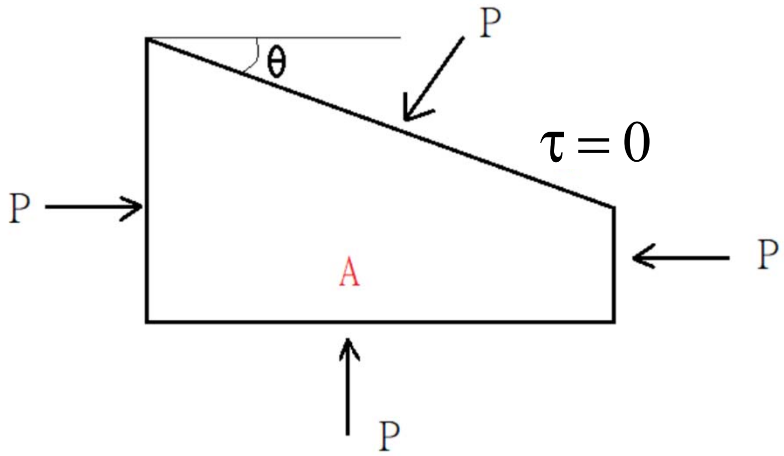
The fluid, as long as the shear stress is applied, moves and deforms continuously.

⇨A fluid at rest must be in a state of zero shear stress.

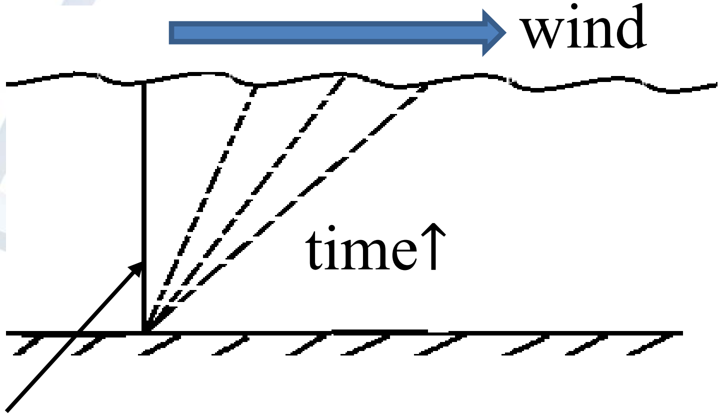
Fluid at rest：



hydrostatic

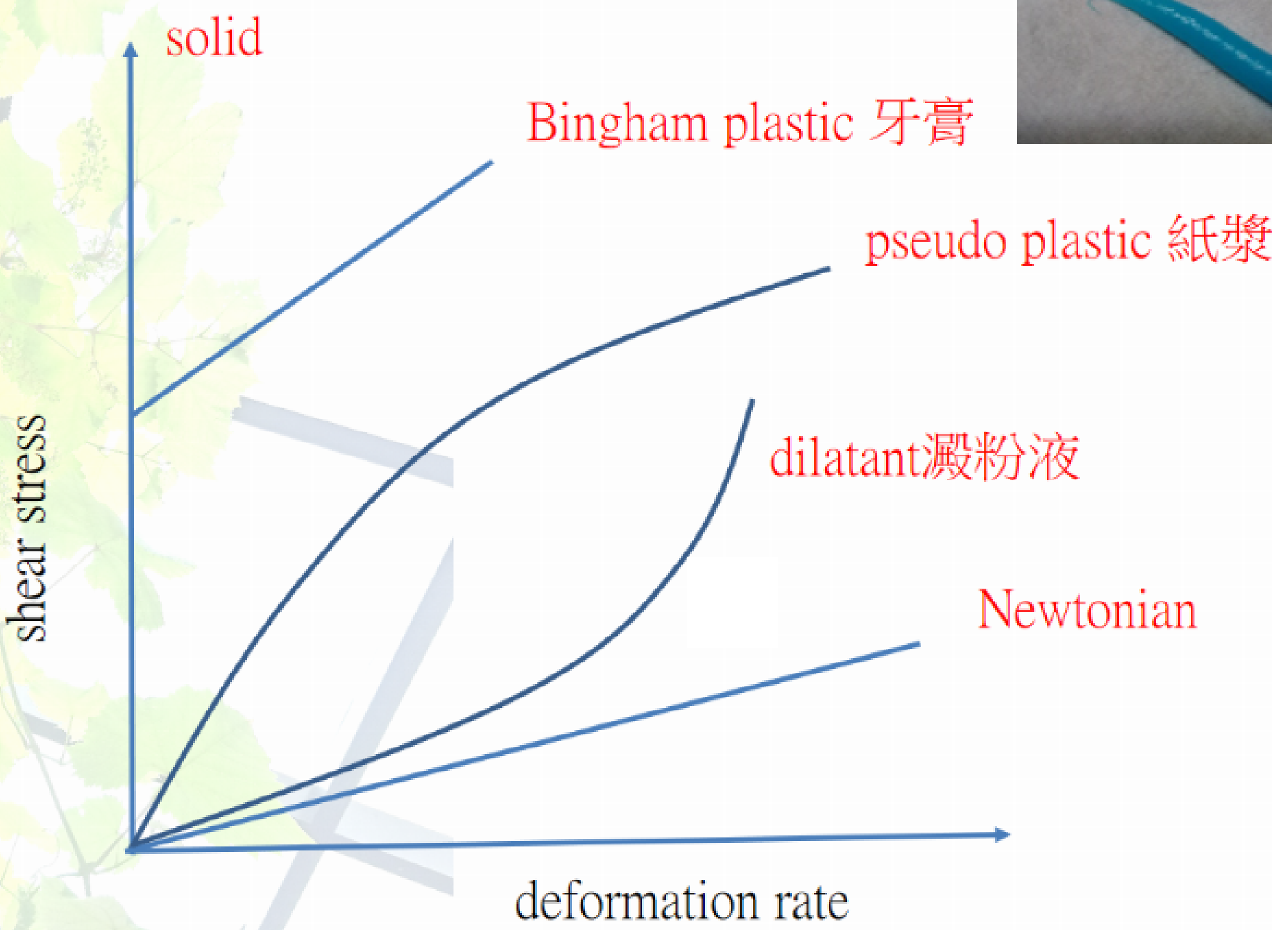


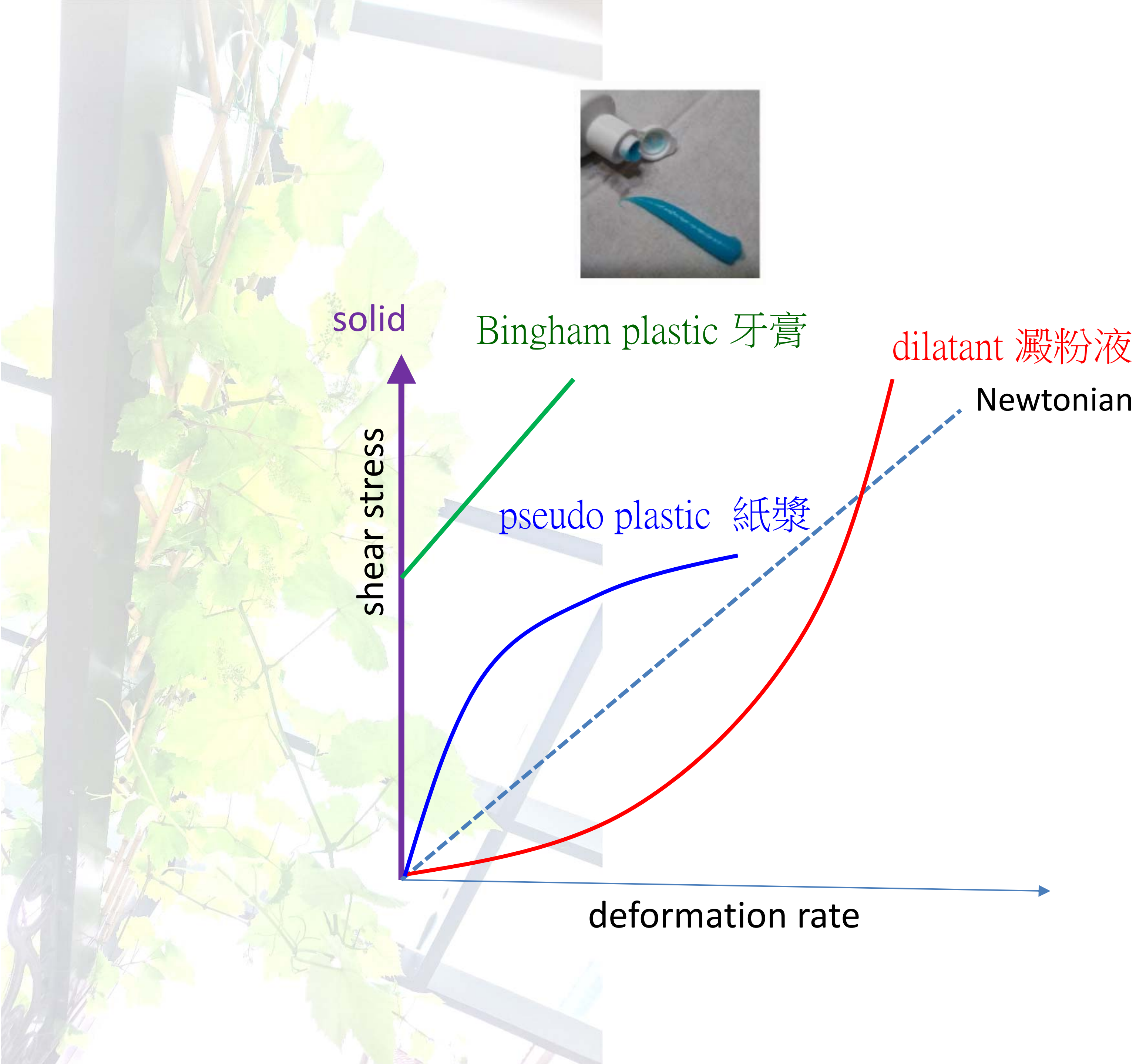
Fluid in motion： Any shear causes motion.



marked particles at time=0

stress
vs
strain rate


solid
Bingham plastic 牙膏
pseudo plastic 紙漿
dilatant澱粉液
Newtonian
shear stress
deformation rate


solid
Bingham plastic 牙膏
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shear stress
pseudo plastic 紙漿
deformation rate

- Fluids cannot hold a shape independent of their surroundings, because of their inability of the intermolecular forces to maintain an unchanging angular orientation of the molecules w.r.t. each other.

- Fluids can be
  mixture, e.g. air, system with chemical reaction (產物 + 反應物)

or

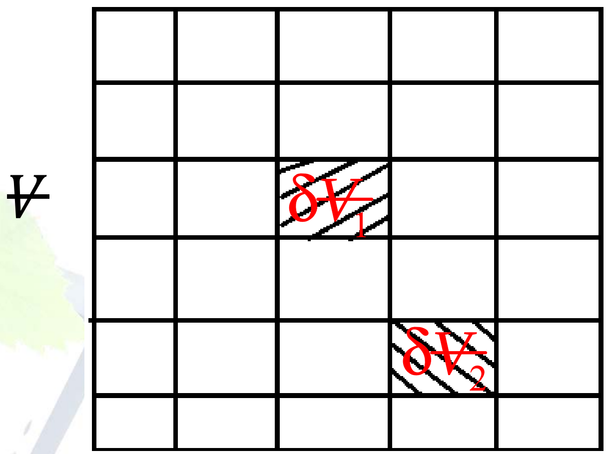
  multiphase, e.g. water + vapor (冷卻循環中之冷媒)

§1.2 continuum

A fluid is called **continuum** which means its variation in properties is so smooth that the differential calculus can be applied.

i.e. fluid properties can be thought of as varying continually in space.

e.g. a container with volume V and total number of molecules N



§1.2 continuum

- ✓ The fluid molecules are in some way randomly distributed in $\forall$. The probabilities for a molecule to located in $\delta\forall_1$ and $\delta\forall_2$ may not be the same.
- ✓ If N is not so large that $(\delta\forall)^{1/3}$ is comparable or less than the molecular spacing or the so-called **mean free path**,

⇨ some $\delta\forall$ have particles, some do not.

each $\delta\forall$ sometimes has and sometimes doesn't have particles.

can not find a ρ representing the density of volume $\delta\forall$ ($\forall$)

⇨ dilute gas (**gas dynamics, molecular dynamics)**

- ✓ If N is so extremely large that the average number of molecules locating in any $\delta\forall$ is relatively large to its fluctuation, then

⇨ one ρ can characterize the density of one $\delta\forall(\vec{x})$.

⇨ continuum

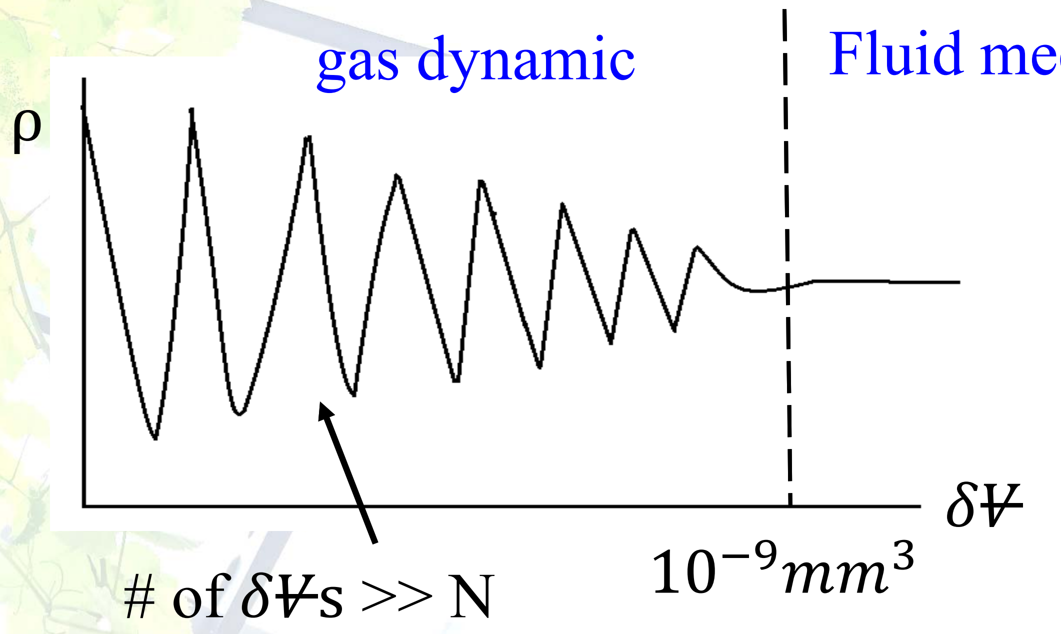
⇨ well defined $\rho(\vec{x}, t)$

§1.2 continuum

Thus, if define $\rho \equiv \frac{m \cdot \delta N}{\delta \forall}$

Where m is the mass of each molecule

δN is the number of molecules found(measured) in one particular $\delta \forall$



Kinetic theory

$$\text{mean free path} = \frac{1}{\sqrt{2}\pi d^2 n_V}$$

$d =$ molecule diameter

n_V = molecules per unit volume

$$= \frac{N_A P}{RT} \text{ for ideal gases}$$

Example: 1atm and 300K : N_2 $(d \approx 0.2nm)$

mean free path

$$= \frac{RT}{\sqrt{2}\pi d^2 N_A P}$$

$$= \frac{8.314\, J/K \cdot mole \times 300K}{\sqrt{2}\pi (0.2nm)^2 \cdot 6\times 10^{23}/mole \cdot 10^5\, N/m^2}$$

=234nm

N_2 at 20°C

	Pressure range	Mean free path (1)	Type of gas flow
Rough vacuum	1000 mbar - 1 mbar	$6.6 \cdot 10^{-8}$ m - $6.6 \cdot 10^{-5}$ m	Viscous flow
Intermediate vacuum	1 mbar - 10^{-3} mbar	$6.6 \cdot 10^{-5}$ m - $6.6 \cdot 10^{-2}$ m	Knudsen flow
High vacuum	10^{-3} mbar - 10^{-7} mbar	$6.6 \cdot 10^{-2}$ m - 660 m	Molecular flow
Ultra high vacuum	< 10^{-7} mbar	> 660 m	Molecular flow

https://helderpad.com/2017/03/02/gas-flow-conductance/

§1.2 continuum

Example: air

$(\delta\mathcal{V})^{1/3} \sim 10^{-6}\, m$ i.e. $\delta\mathcal{V} \sim 10^{-18}\ m^3$

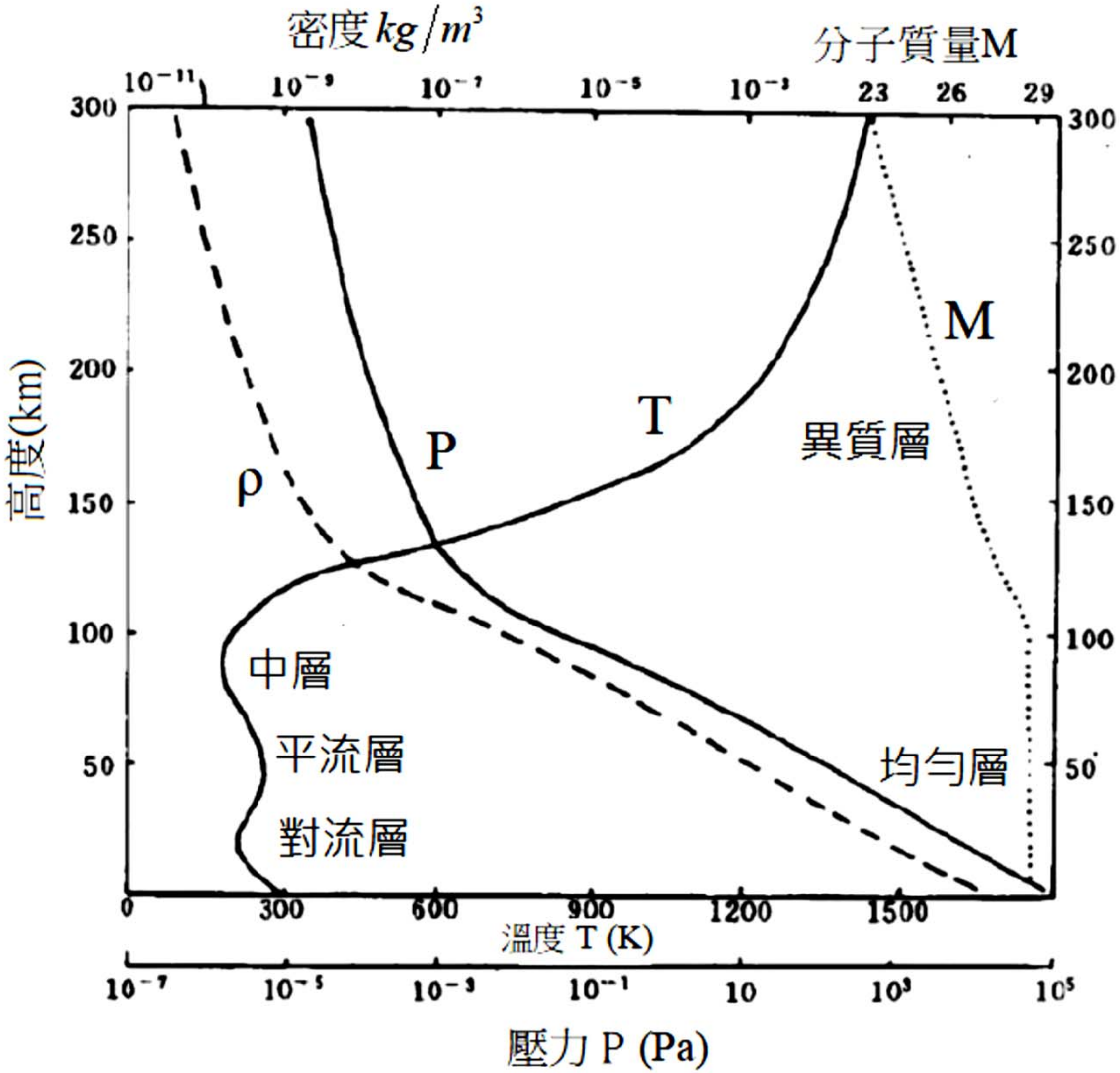
@STR: total N~ 10^7 >>1

In fluid mechanics, $\rho \equiv \lim_{\delta\mathcal{V}\to 0} \frac{\delta m}{\delta\mathcal{V}} = \rho(x,y,z,t)$

in such a way that there are still many enough molecules in $\delta\mathcal{V}$

Fluid mechanics is a **macroscopic** science.

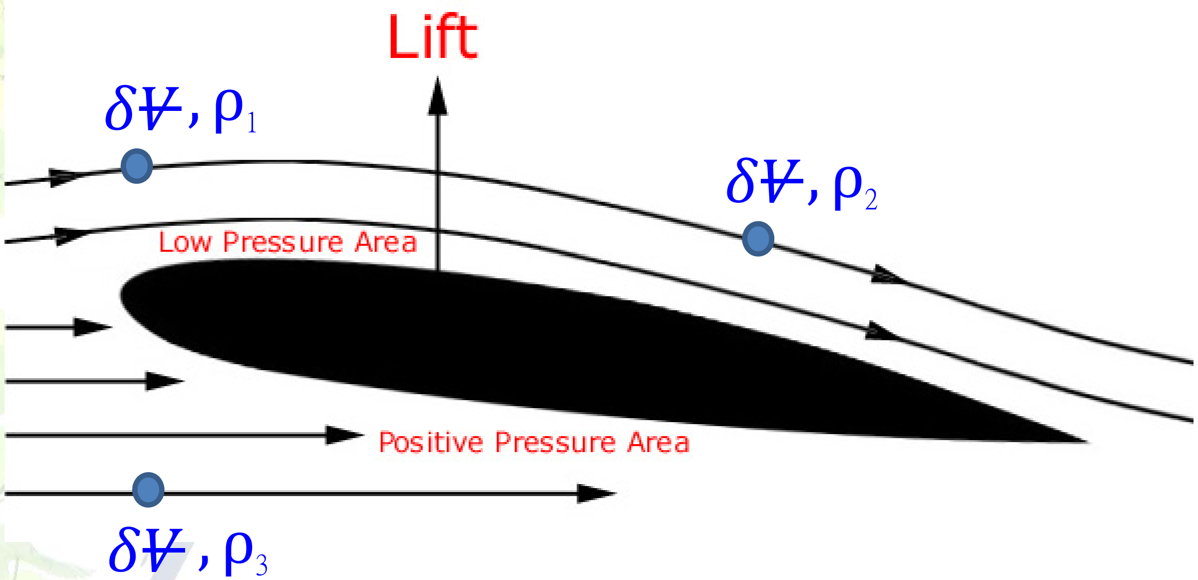
§1.2 continuum



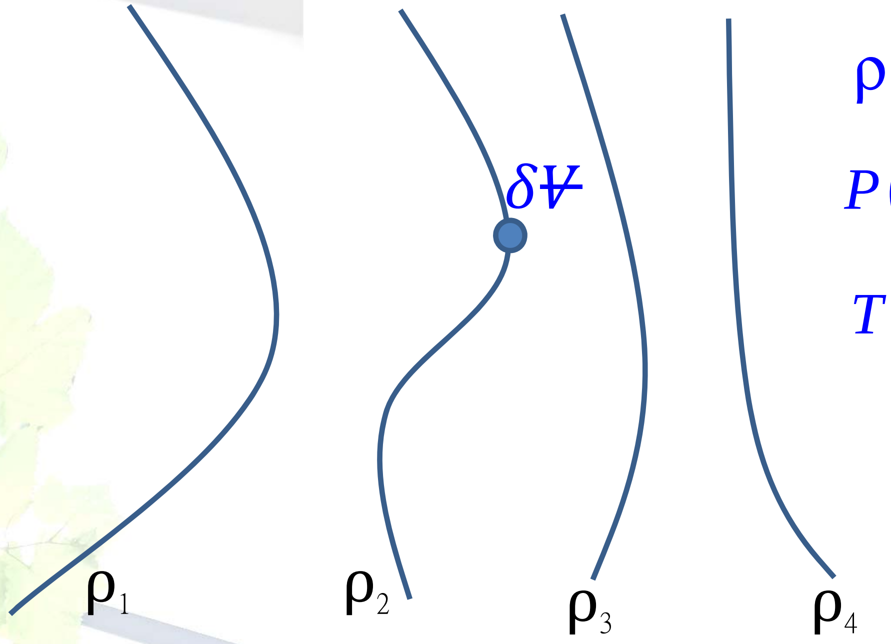
https://read01.com/ePJN06O.html#.YxxRbHZBy5c
來源：熊年祿等《電離層物理概論》

§1.2 continuum

- Study the average behavior of a very large number of molecules in the vicinity of a point in a fluid.
- It is concerned with characteristics that can be observed and measured on the laboratory scale.



§1.2 continuum



$\rho(x,y,z,t)$

$P(x,y,z,t)$

$T(x,y,z,t)$

- A fluid particle is defined as a small mass of fluid of fixed identity of volume $\delta V \sim 10^{-9} mm^3$.
- Thermodynamic Properties: Assume all timescales and length scales involved with the molecular motions are much smaller than the laboratory scales. (e.g. collision time, mean free path etc.) so that a fluid subjected to sudden changes rapidly adjusts itself toward equilibrium.

(**local thermodynamic equilibrium**)

§1.2 continuum

- Thermodynamic properties exist as point functions and follow all the laws and state relation of ordinary equilibrium thermodynamics (such as *PV*=*nRT*) .

- Fluid velocity $\vec{u}(x,y,z,t)$ is the mean velocity of molecules within $\delta \forall$ which instantaneously surrounding point Q(*x*,*y*,*z*).

$\rho(x,y,z,t)$ density field

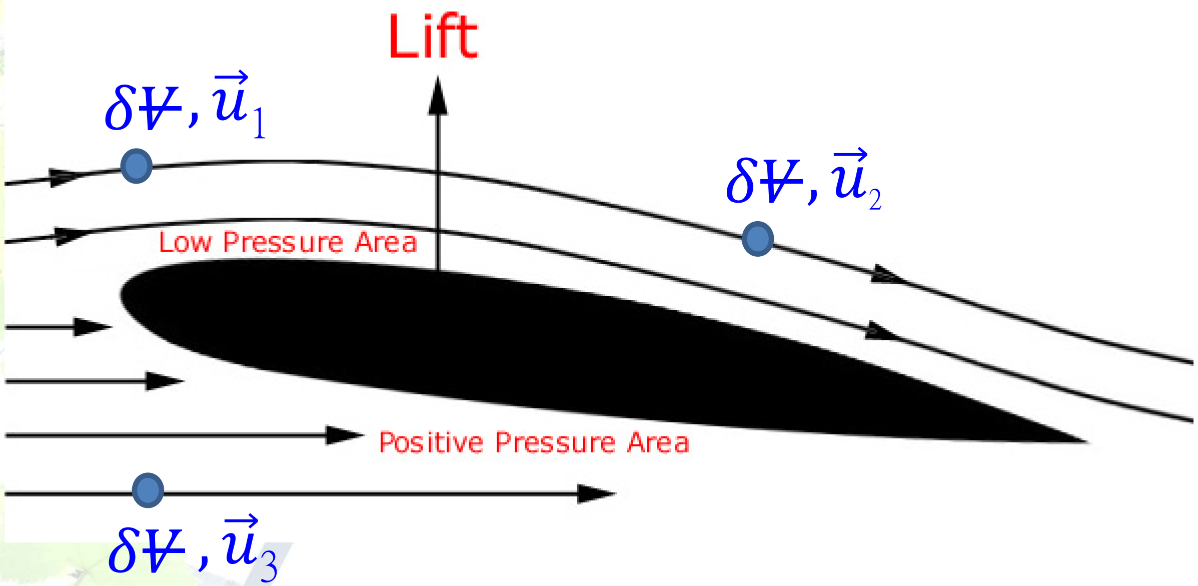
$P(x,y,z,t)$ pressure field

$T(x,y,z,t)$ temperature field

$\vec{u}(x,y,z,t)$ velocity field

§1.2 continuum

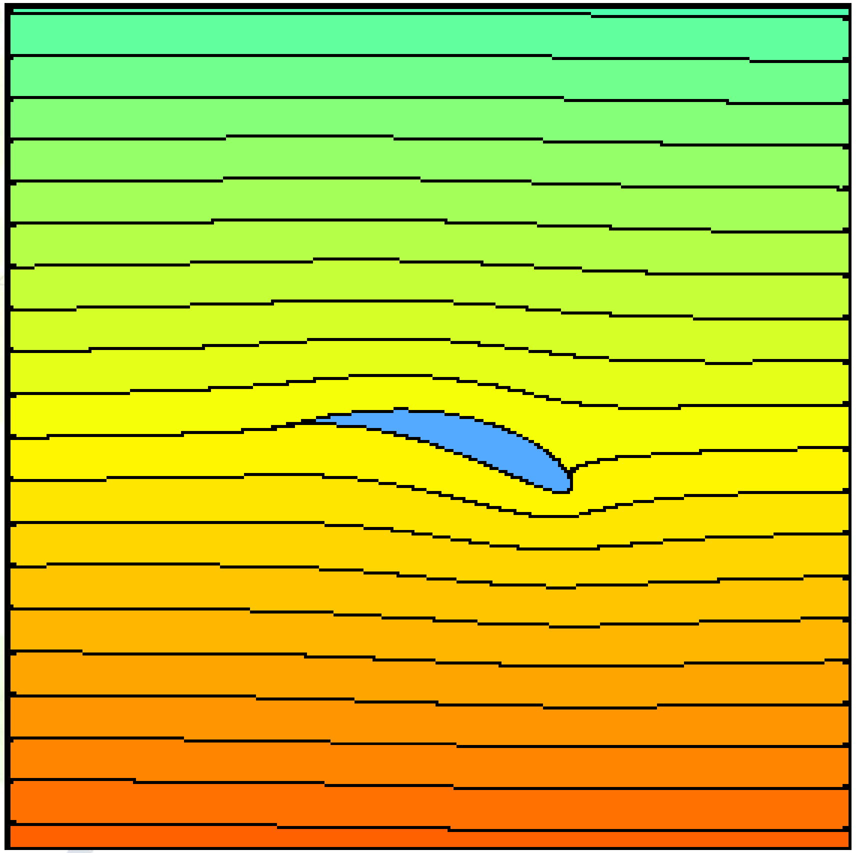
Streamline: a curve tangential to the velocity vector everywhere



https://www.grc.nasa.gov/WWW/K-12/airplane/foil3.html

§1.3 flowlines

Steamline: a curve tangential to velocity vector everywhere



https://www.av8n.com/irro/profilo1_e.html

§1.3.1 streamlines

A **streamline** in a flow field that is everywhere tangent to the velocity for **any instant of time *t***.

- ✓ No flow can cross a streamline.
- ✓ Streamlines may change in time.

$$\vec{x}_2 = \vec{x}(s+ds) = \vec{x}_1 + d\vec{x} = \vec{x}(s) + d\vec{x}$$

$$d\vec{x} = (dx, dy, dz) \parallel \vec{u} = (u, v, w)$$

$$\vec{u} \times d\vec{x} = 0 \quad \Rightarrow \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \equiv ds$$

parameter: s

$$\frac{dx}{ds} = u(x, y, z, t)$$

$$\frac{dy}{ds} = v(x, y, z, t)$$

$$\frac{dz}{ds} = w(x, y, z, t)$$

IC: $(x, y, z) = (x_0, y_0, z_0)$ at $s = 0$

§1.3.1 streamlines

Example: $\vec{u}=\left(2x,-yt\right)$

$$\frac{dx}{ds}=2x \;\Rightarrow\; x=x_0e^{2s}$$

parameter = *s*

$$\frac{dy}{ds}=-yt \;\Rightarrow\; y=y_0e^{-ts}$$

$$\left(\frac{x}{x_0}\right)^t\left(\frac{y}{y_0}\right)^2=1$$

Given $t,x_0,y_0 \Rightarrow y(s)=y\left(x(s)\right)$

e.g. $\left(x_0,y_0,t\right)=(2,1,4)$

$$x^4y^2=16 \;\Rightarrow\; x^2y=4$$

$$\vec{u}\times d\vec{s}=0$$

$$\left(2x,-yt,0\right)\times\left(dx,dy,0\right)=0$$

$$\left(2xdy+ytdx\right)\vec{e}_z=0$$

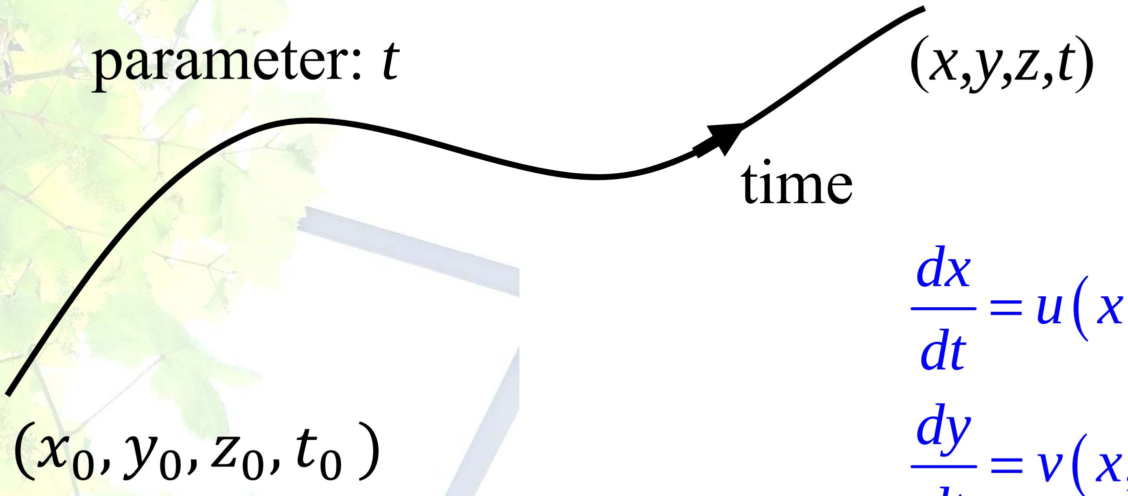
$$\frac{dx}{2x}=-\frac{dy}{yt}$$

$$\frac{1}{2}\ln\left(\frac{x}{x_0}\right)=-\frac{1}{t}\ln\left(\frac{y}{y_0}\right)$$

$$\left(\frac{x}{x_0}\right)^t\left(\frac{y}{y_0}\right)^2=1$$

§1.3.2 pathlines

A **pathline** is the path or trajectory traced out by a particular fluid particle.



$$\frac{dx}{dt} = u(x,y,z,t)$$

$$\frac{dy}{dt} = v(x,y,z,t)$$

$$\frac{dz}{dt} = w(x,y,z,t)$$

Given (Lagrangian marker)

$$\vec{x} = \vec{x}(t; t_0, \vec{x}_0)$$

parameter

IC: $(x,y,z) = (x_0, y_0, z_0)$ at $t = t_0$

§1.3.2 pathlines

Example: $\vec{u} = (2x, -yt)$

$$\frac{dx}{dt} = u = 2x$$

$$\frac{dy}{dt} = v = -yt$$

$$\Rightarrow \frac{dx}{x} = 2dt \;\Rightarrow\; \ln\left(\frac{x}{x_0}\right) = 2(t - t_0)$$

$$\Rightarrow \frac{dy}{y} = -tdt$$

$$\Rightarrow \ln\left(\frac{y}{y_0}\right) = -\frac{1}{2}\left(t^2 - t_0^2\right)$$

$$\Rightarrow x(t) = x_0 \exp\left[2(t - t_0)\right]$$

$$\Rightarrow y(t) = y_0 \exp\left[-\frac{1}{2}\left(t^2 - t_0^2\right)\right]$$

~ parametric form

$$t = t_0 + \frac{1}{2}\ln\left(\frac{x}{x_0}\right)$$

$$\ln\left(\frac{y}{y_0}\right) = -\frac{1}{2}\left\{\left[t_0 + \frac{1}{2}\ln\left(\frac{x}{x_0}\right)\right]^2 - t_0^2\right\}$$

Given $t_0, x_0, y_0 \Rightarrow y(t) = y(x(t))$

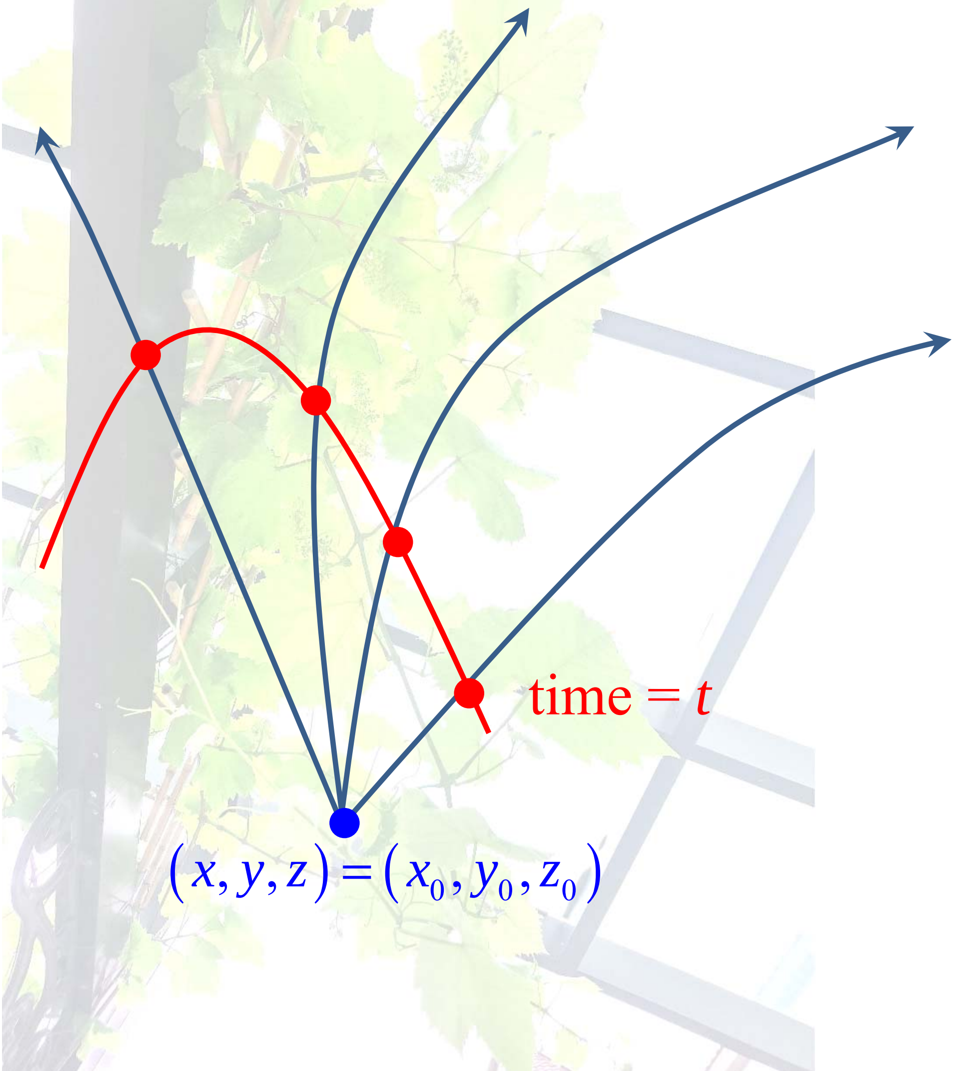
§1.3.3 streaklines

A **streakline** is a line in a flow field which is the locus of particles which have earlier passed through a prescribed point.

given

$$\vec{x} = \vec{x}\left(t; t_0, \vec{x}_0\right)$$

parameter



P1: $\left(x, y, z\right) = \left(x_1, y_1, z_1\right)$ at time $= t$

P2: $\left(x, y, z\right) = \left(x_2, y_2, z_2\right)$ at time $= t$

P3: $\left(x, y, z\right) = \left(x_3, y_3, z_3\right)$ at time $= t$

⋮

P1: $\left(x, y, z\right) = \left(x_0, y_0, z_0\right)$ at time $= t_{01}$

P2: $\left(x, y, z\right) = \left(x_0, y_0, z_0\right)$ at time $= t_{02}$

P3: $\left(x, y, z\right) = \left(x_0, y_0, z_0\right)$ at time $= t_{03}$

⋮

§1.3.3 streaklines

Example: $\vec{u} = (2x, -yt)$

$$\ln\left(\frac{x}{x_0}\right) = 2(t - t_0)$$

$$\ln\left(\frac{y}{y_0}\right) = -\frac{1}{2}\left(t^2 - t_0^2\right)$$

streakline: parameter = t_0

Given $t, x_0, y_0 \Rightarrow y(t_0) = y\left(x(t_0)\right)$

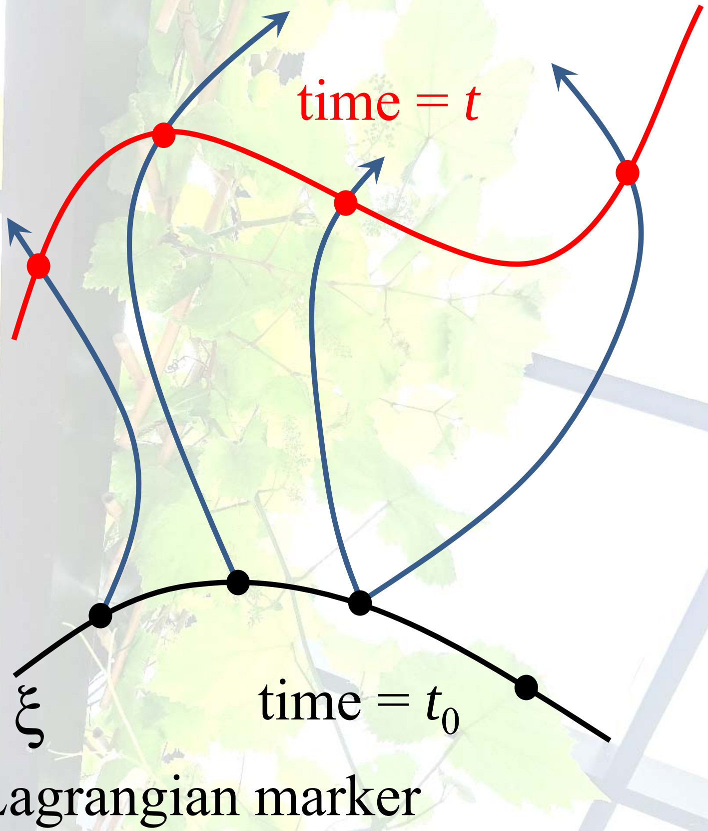
$$\ln\left(\frac{y}{y_0}\right) = -\frac{1}{2}\left\{t^2 - \left[t - \frac{1}{2}\ln\left(\frac{x}{x_0}\right)\right]^2\right\}$$

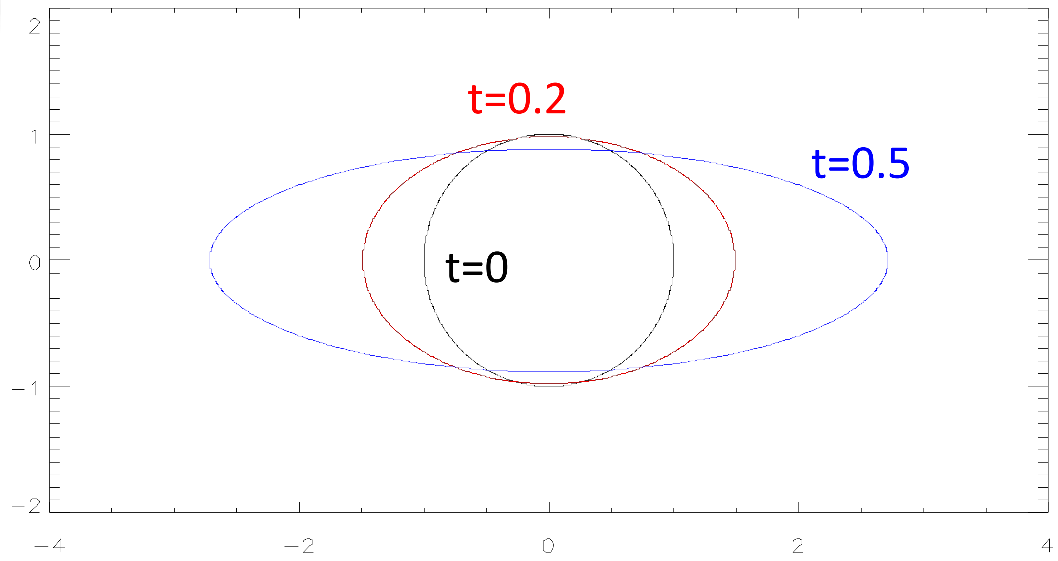
Pathline: parameter = t

Given $t_0, x_0, y_0 \Rightarrow y(t) = y\left(x(t)\right)$

$$\ln\left(\frac{y}{y_0}\right) = -\frac{1}{2}\left\{\left[t_0 + \frac{1}{2}\ln\left(\frac{x}{x_0}\right)\right]^2 - t_0^2\right\}$$

§1.3.4 material lines





Material line: parameter = ξ

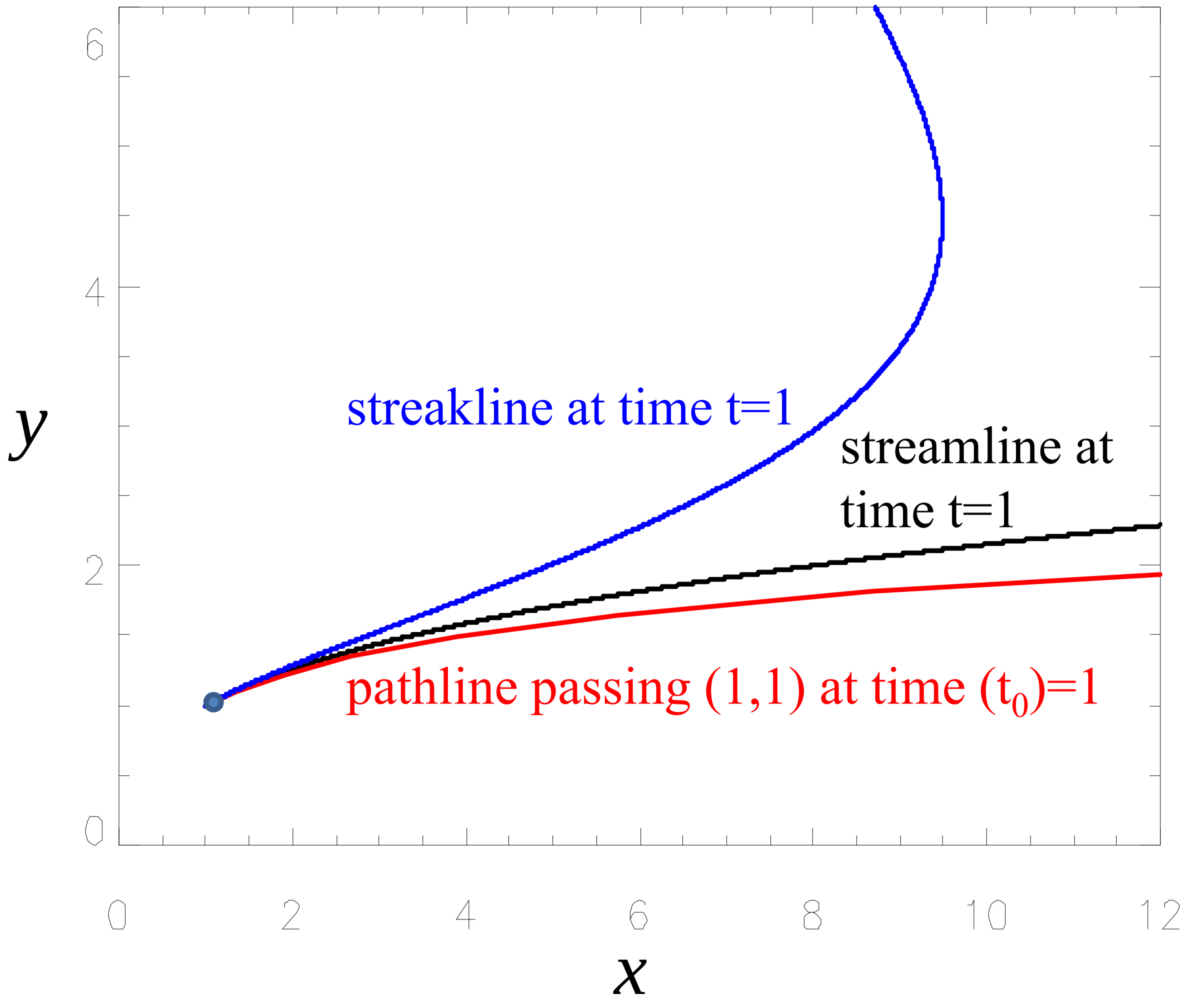
$$x = x_0(\xi)\exp\left[2(t-t_0)\right]$$

$$y = y_0(\xi)\exp\left[-\frac{1}{2}\left(t^2-t_0^2\right)\right]$$

Given $t, t_0, \{x_0(\xi), y_0(\xi)\} \Rightarrow y(\xi) = y(x(\xi))$

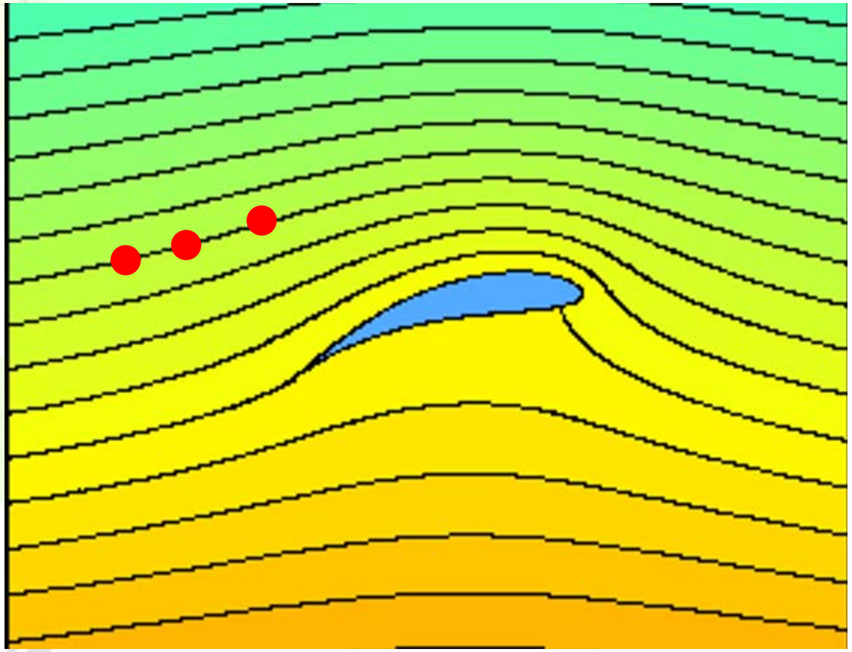
Example:

$$u = x(1+2t)$$
$$v = y$$



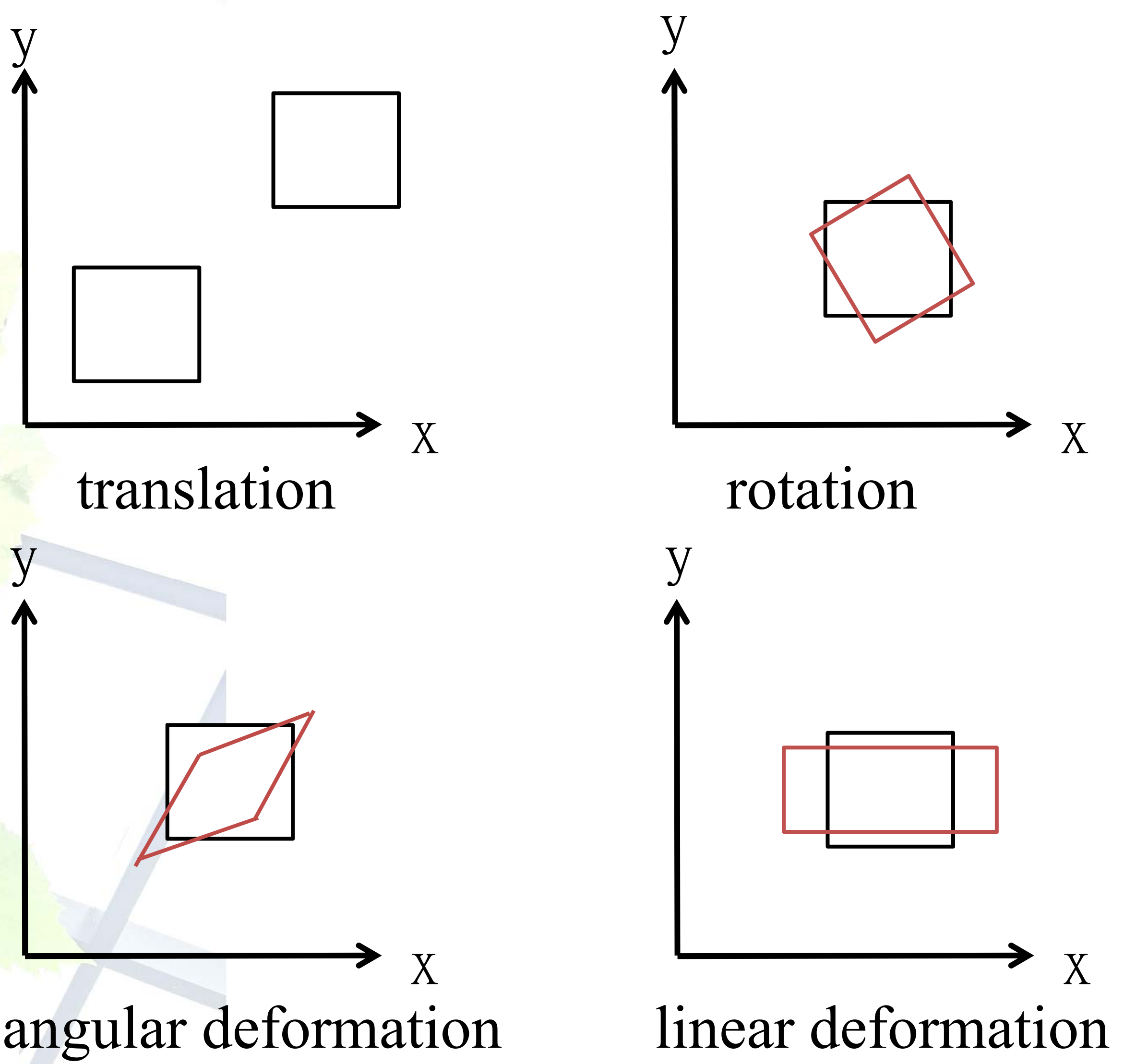
Steady flows

~ time-independent fields
~ A streamline, pathline, streakline passing through a same reference point correspond to a same curve.



https://www.av8n.com/irro/profilo1_e.html

1.4 fluid motion



motion= translation + solid rotation + deformation

1.4.1 Strain rate

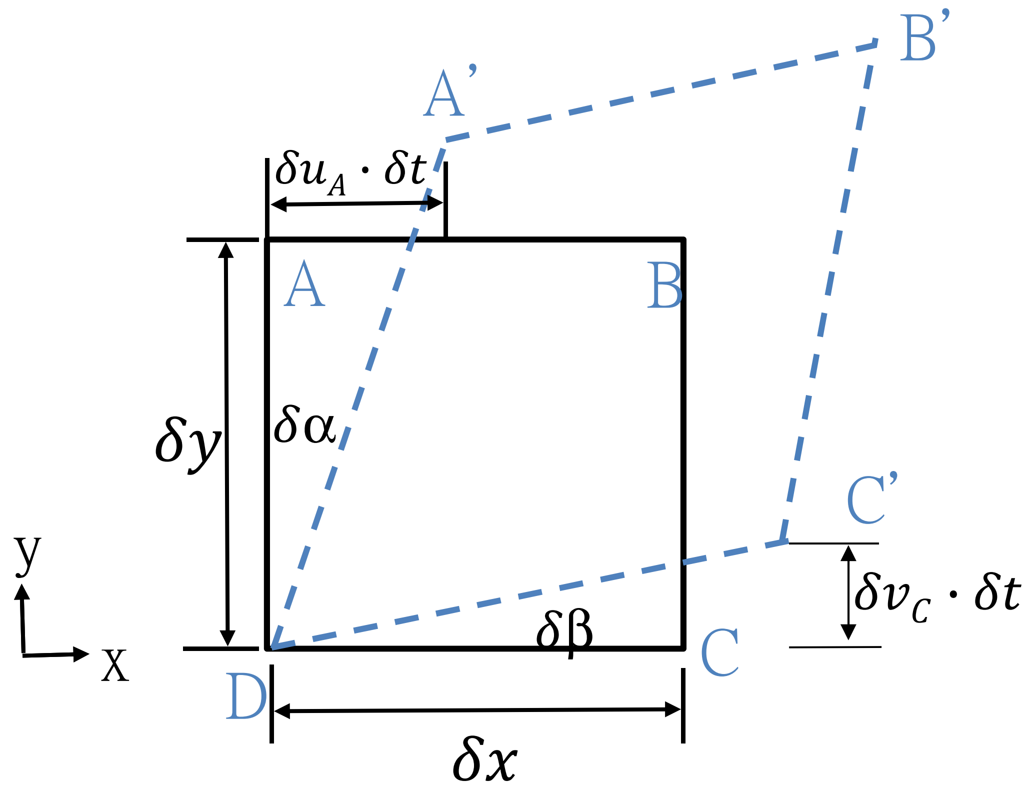
Consider a fluid element

$$u_A = u + \frac{\partial u}{\partial y}\delta y$$

$$v_A = v + \frac{\partial v}{\partial y}\delta y$$

$$u_C = u + \frac{\partial u}{\partial x}\delta x$$

$$v_C = v + \frac{\partial v}{\partial x}\delta x$$



$$\vec{x}_D = (x, y)$$

$$\vec{x}_A = (x, y + \delta y)$$

$$\vec{x}_C = (x + \delta x, y)$$

$$\delta\alpha \approx \tan\delta\alpha = \frac{\delta u_A \cdot \delta t}{\delta y + \delta v_A \cdot \delta t} = \frac{\frac{\partial u}{\partial y}\delta y \cdot \delta t}{\delta y} = \frac{\partial u}{\partial y} \cdot \delta t$$

$$\delta\beta \approx \tan\delta\beta = \frac{\delta v_C \cdot \delta t}{\delta x + \delta u_C \cdot \delta t} = \frac{\frac{\partial v}{\partial x}\delta x \cdot \delta t}{\delta x} = \frac{\partial v}{\partial x} \cdot \delta t$$

1.4.1 Strain rate

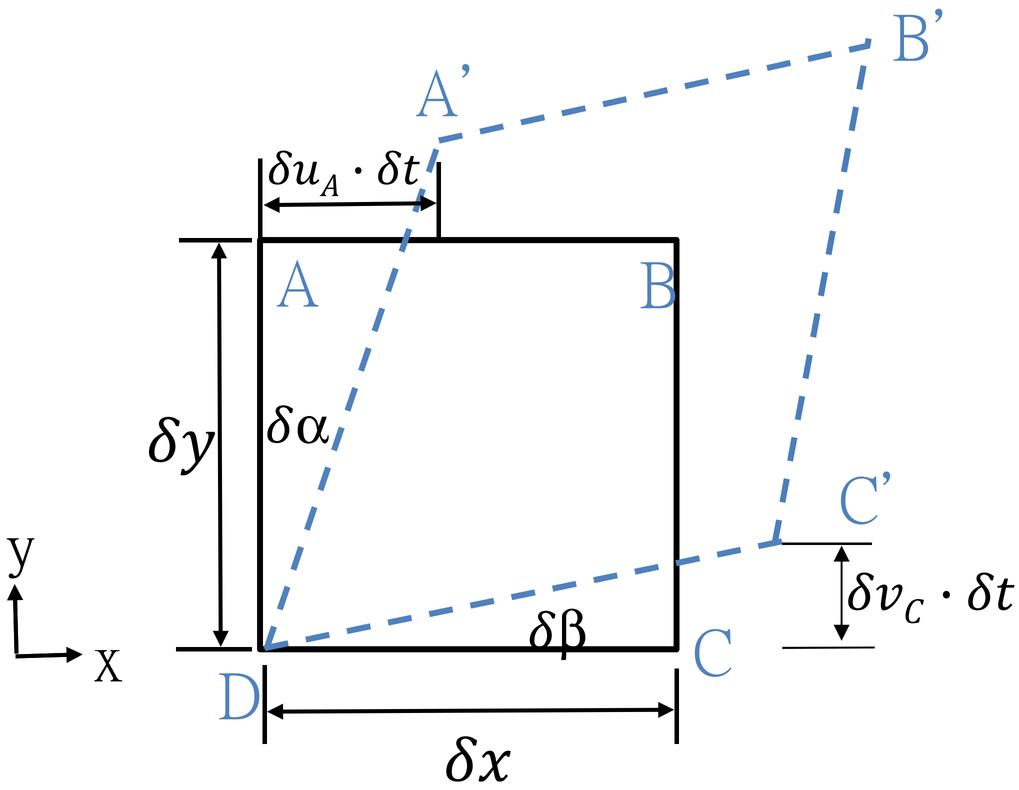
Consider a fluid element

Strain rate:

$$S = \frac{1}{2}\frac{(\delta\beta + \delta\alpha)}{\delta t} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

Rotational rate:

$$\Omega = \frac{1}{2}\frac{\delta\beta - \delta\alpha}{\delta t} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$



$$\delta\alpha = \frac{\partial u}{\partial y}\cdot\delta t$$

$$\delta\beta = \frac{\partial v}{\partial x}\cdot\delta t$$

1.4.2 Stress

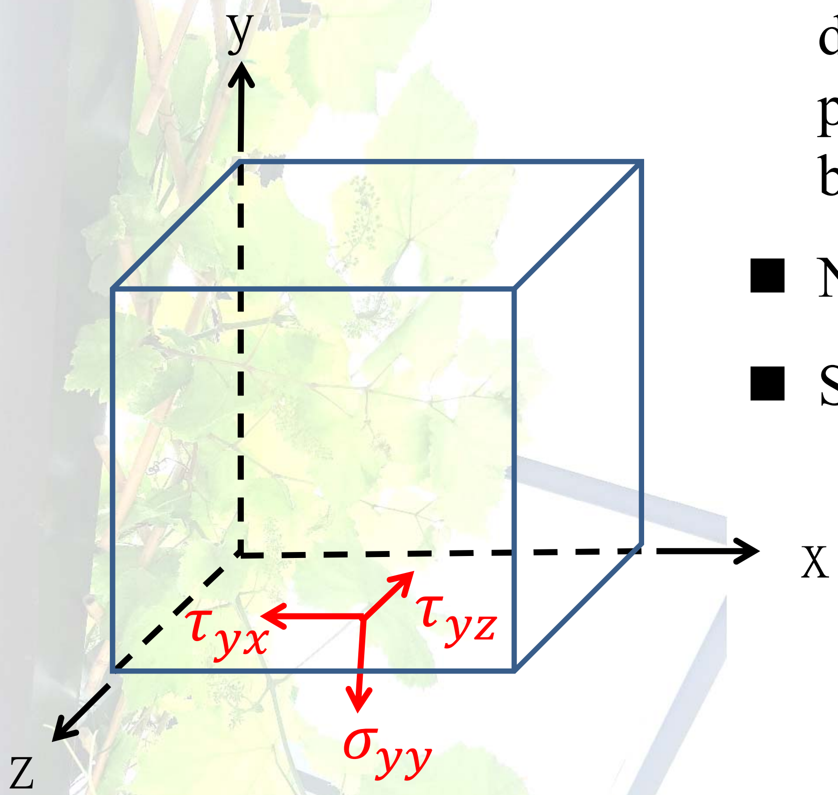
Stress $\tau_{xy} = \lim_{\delta A_x \to 0} \frac{\delta F_y}{\delta A_x}$

first subscript： the normal direction of the plane on which the stress acts

second subscript ：the direction in which the stress acts

the state of stress at a point ： $\begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix}$

1.4.2 Stress

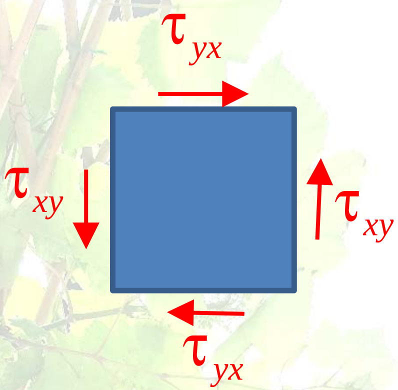


- A stress components is positive when the direction of the stress component and the plane on which it acts are both positive or both negative.
- Normal stress: σ_{xx}, σ_{yy}, σ_{zz}
- Shear stress: τ_{xy}, τ_{yz}, τ_{zx}, τ_{yx}, τ_{zy}, τ_{xz}

Surface forces (stress): the force acting between molecules on the surface and molecules outside the fluid particle in the surrounding medium, i.e. intermolecular forces.

Shear stress causes continuous shear deformation in a fluid.

1.4.2 Stress Symmetry $\tau_{xy} = \tau_{yx}$



$$\text{torque} = 2 \cdot \frac{\delta x}{2} \cdot \left(\tau_{xy} \delta y \right) - 2 \cdot \frac{\delta y}{2} \cdot \left(\tau_{yx} \delta x \right)$$

$$= \text{inertial moment} \cdot \frac{d^2 \theta}{dt^2}$$

$$\text{inertial moment} \sim \rho \delta x \delta y \cdot \left(\delta x^2 + \delta y^2 \right)$$

As $\delta x, \delta y \to 0$: $$2 \cdot \frac{\delta x}{2} \cdot \left(\tau_{xy} \delta y \right) - 2 \cdot \frac{\delta y}{2} \cdot \left(\tau_{yx} \delta x \right) = 0$$

$$\tau_{xy} = \tau_{yx}$$

1.4.3 Newtonian fluids

A **Newtonian fluid** is one where there is a **linear** relationship between stress and strain-rate. E.g. **water, air , gasoline** under normal condition.

$$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \tau_{yx}$$

$$\sigma_{xx} = -\left(P - \lambda \nabla \cdot \vec{u}\right) + 2\mu \frac{\partial u}{\partial x}$$

μ is called the **shear viscosity** coefficient.

λ is called the **second viscosity**.

$\kappa = 2\mu/3 + \lambda$ is called the **bulk viscosity (=0, Stokes' hypothesis)**.

$\kappa = 0$ for dilute monatomic gases

$\kappa \approx 3\mu$ for water

negligible unless volume expansion is huge.

$\lambda \approx -2\mu/3$

1.4.3 Newtonian fluids

$$P_m \equiv -\frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3} = P - \left(\lambda + \frac{2\mu}{3}\right)(\nabla \cdot \vec{u}) \qquad \text{(mechanical pressure)}$$

P : thermodynamic pressure

$\kappa = \lambda + \frac{2\mu}{3} \geq 0$ by thermodynamic second law

- Shear viscosity μ strongly depends on temperature

  $\mu \uparrow$ as T $\uparrow$ gasses

  $\mu \downarrow$ as T $\uparrow$ liquid

- weakly depends on pressure

- Kinetic viscosity (momentum diffusivity) $\nu = \mu/\rho$

1.4.4 non-Newtonian fluids

$$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

Newtonian fluids: μ = constant

Non-Newtonian fluids： mostly due to very large fluid molecules

- **dilatant**：deformation rate ↑ ⇨ μ ↑

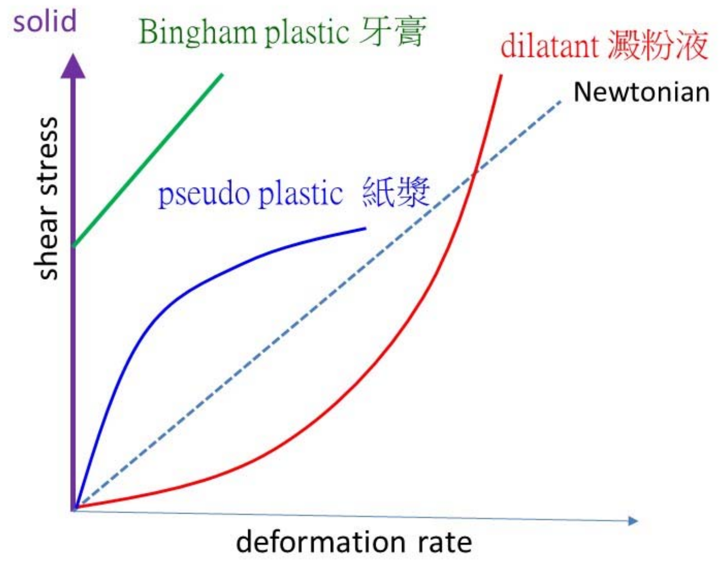
  e.g. 澱粉懸浮夜、砂粒懸浮夜

- **pseudo plastic**：deformation rate ↑ ⇨ μ ↓

  e.g. polymer solution 、紙漿

- **Bingham plastic**：behaves like a solid when the shear stress is less than some yielding stress; behaves like a fluid thereafter

  e.g. 牙膏



https://www.youtube.com/watch?v=G1Op_1yG6lQ

1.4.4 non-Newtonian fluids

- **thixotropic**： μ↓as time↑which shear stress keeps constant.

  e.g. 油漆

- **rheopectic**： μ↑as time↑

  https://www.youtube.com/watch?v=S8gP3yWsloc

- **viscoelastic**：fluids partially return to their original shape after the shear stress is released.

Remark: viscosity~ molecular interactions
~ lead to viscous drag (τ_{xy})
~ cause momentum transfer

Unit of viscosity

$$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$[\mu] = \frac{s \cdot N}{m^2} = \frac{s \cdot kg \cdot m/s^2}{m^2} = \frac{kg}{s \cdot m}$$

$$\left[\tau_{xy}\right] = \frac{N}{m^2}$$

$$[\nu] = \left[\frac{\mu}{\rho}\right] = \frac{kg}{s \cdot m} \cdot \frac{m^3}{kg} = \frac{m^2}{s}$$

$$\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right] = \frac{1}{s}$$

e.g. 1 atm, 20°C

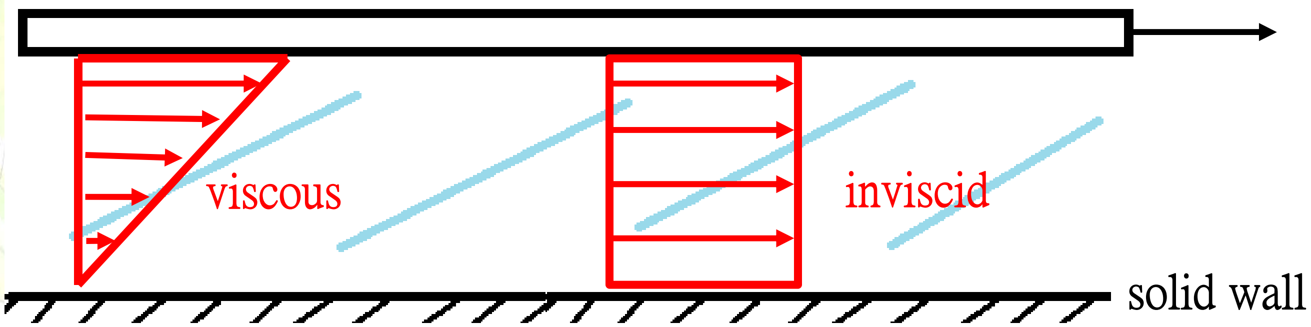
air	$\mu = 1.8 \cdot 10^5\, kg/m \cdot s$	$\nu = 1.51 \cdot 10^5\, m^2/s$
water	$\mu = 10^{-3}\, kg/m \cdot s$	$\nu = 1.01 \cdot 10^{-6}\, m^2/s$
mercury	$\mu = 1.5 \cdot 10^{-3}\, kg/m \cdot s$	$\nu = 1.16 \cdot 10^{-7}\, m^2/s$

1.4.5 Inviscid flow vs Viscous flow

- inviscid flow: $\mu = 0$, no inter-molecular forces
- inviscid fluids do not exist; all fluids posses viscosity
- the assumption of $\mu = 0$ can simplify analysis and get meaningful results.
- In any viscous flow, the fluid in contact with a solid boundary has the same velocity as the boundary itself.

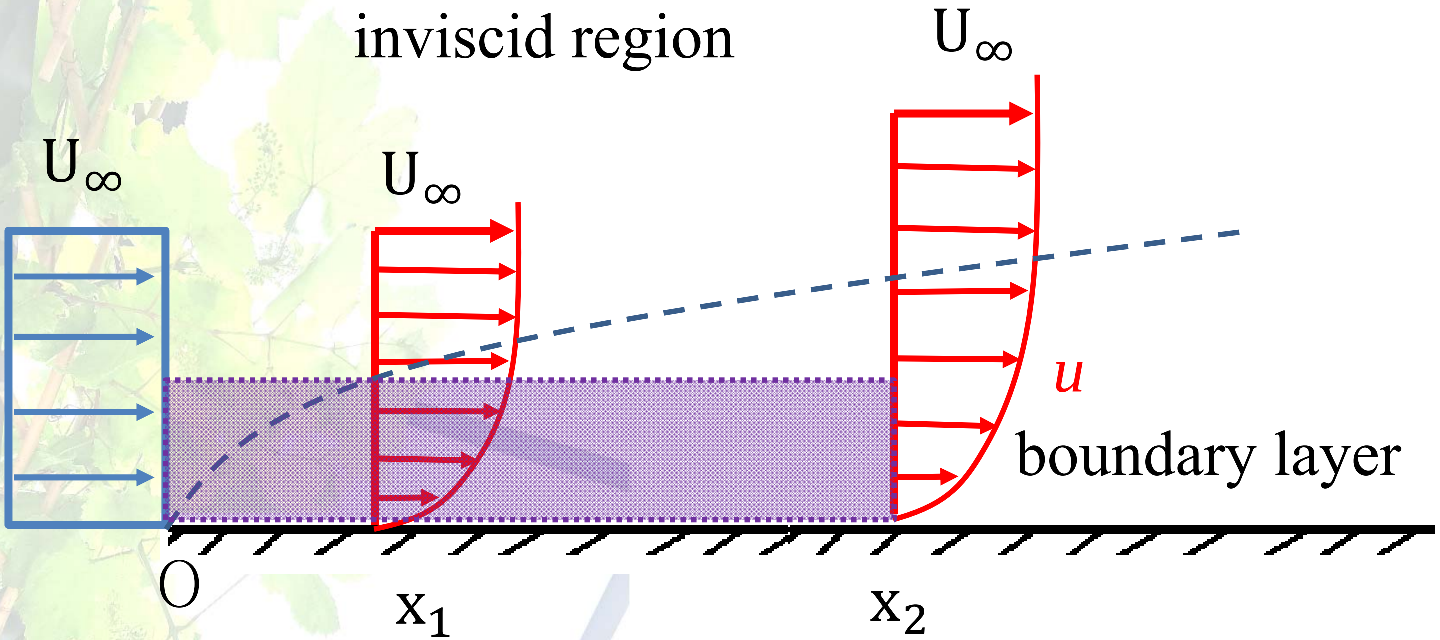
~ **nonslip boundary condition**

fluids at the belt has the same velocity as that of the belt (plate)



flows at wall have zero velocity

1.4.5 Viscous flow

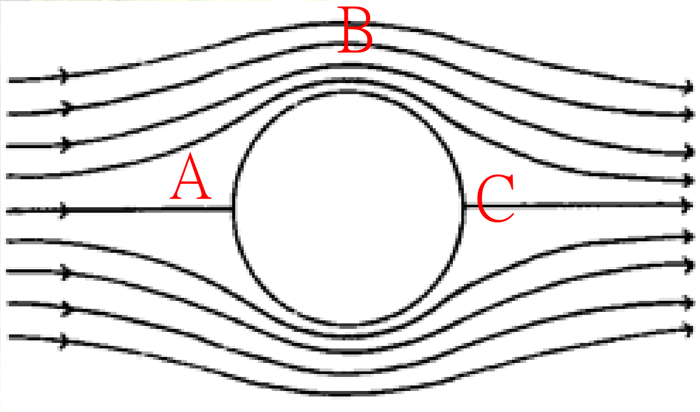


Streamlines parallel to the plate?

No! $v > 0$ for mass conservation.

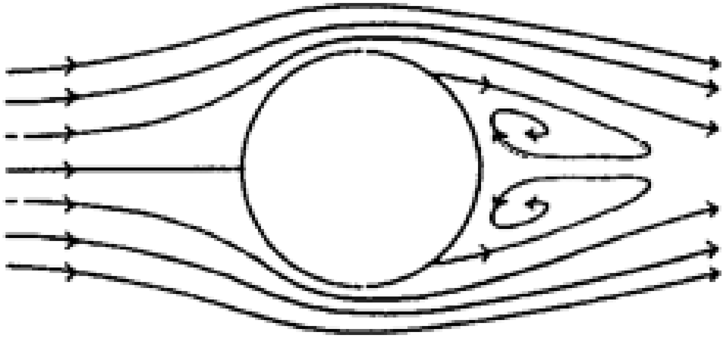
1.4.5 Inviscid flow vs Viscous flow

inviscid



viscous

separation



Viscous

total drag = pressure drag

+ viscous drag

Inviscid

A: stagnation point

- velocity ↑ from A to B; ↓ from B to C
- pressure ↓ from A to B; ↑ from C to B
- symmetry ⇨ no pressure drag
- inviscid ⇨ no shear stress ⇨ no viscous drag

1.4.5 Viscous flow

adverse pressure gradient

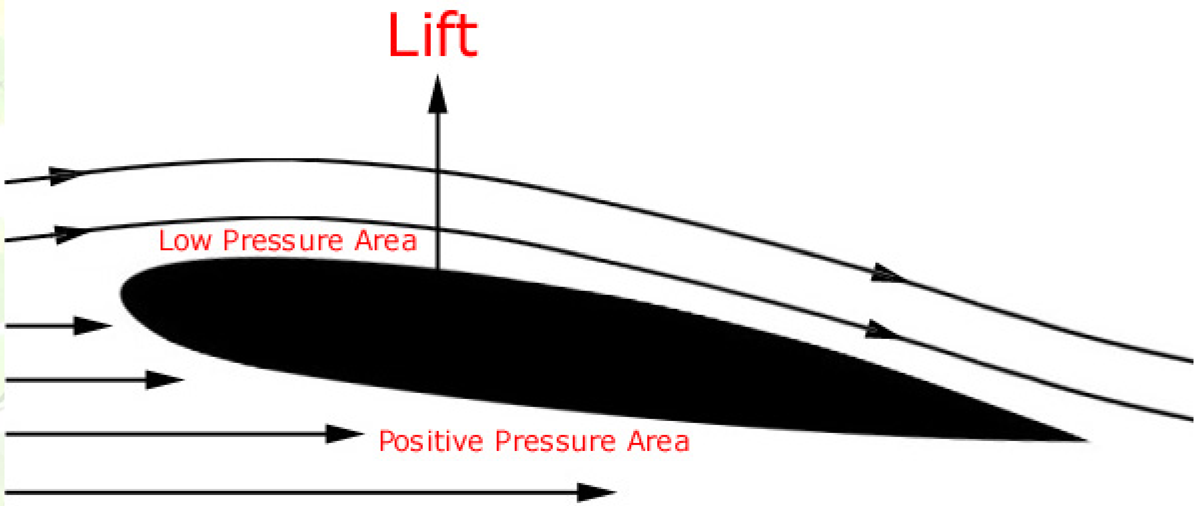
"streamlining " shape ⇨ reduce adverse pressure gradient

⇨ delay the separation

⇨ reduce pressure drag

⇨ viscous drag increases (∵surface increases)

⇨ net drag reduced



1.5 Dimensional Analysis

Buckingham Pi Theorem

~ give suggestions for possible grouping of related parameters such that the groups of parameters, not the parameters themselves, are the key factors determining the behaviors of the given system.

dimensions and units

- A dimension is the measure by which physical variable is expressed quantitatively,

  e.g. length, time, temperature, force, torque,……

- A unit is a particular way of attaching a number to a dimension

  e.g. force: $[F]$ Newton, $kg \cdot m/s^2$, lbf, …

  time: $[t]$ second, minute, hour, day, …

- Primary dimensions: those dimensions which basically express all observable physical quantities and are *independent* from each other (none of them be measured in terms of any combination of the others).

e.g. {mass, time, length, temperature, electric field} $=\{M, t, L, T......\}$

or {force, time, length, temperature, electric field} $=\{F, t, L, T......\}$

1.5 Dimensional Analysis - Buckingham Pi Theorem

Given a physical problem and

$$q_1 = f(q_2, q_3 \ldots\ldots, q_n) \qquad \text{or} \qquad F(q_1, q_2 \ldots\ldots, q_n) = 0$$

q_1: dependent variable; $q_2, q_3 \ldots\ldots, q_n$: n-1 indep. variables

- Let ***m*** be the minimum number of *independent dimensions* required to specify the dimensions of all the parameters $q_1, q_2 \ldots\ldots, q_n$.
- Then these ***n*** parameters can be grouped into ***n-m*** independent dimensionless parameters, Π parameters, such that

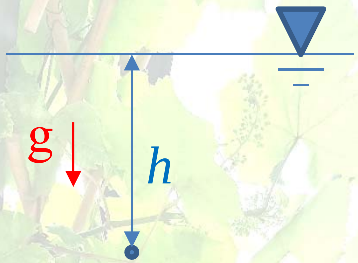
$$\Pi_1 = f(\Pi_2, \Pi_3, \ldots\ldots, \Pi_{n-m})$$

or

$$F(\Pi_1, \Pi_2, \Pi_3, \ldots\ldots, \Pi_{n-m}) = 0$$

~ requirement of consistency of dimension ~

Example 1:



$$\Delta P = P - P_{atm} = f(\rho, g, h)$$ n=4

$$[\Delta P] = \left[\frac{F}{L^2}\right] = \frac{N}{m^2} = \frac{kg \cdot m/s^2}{m^2} = \frac{kg}{m \cdot s^2}$$

$$[\rho] = \left[\frac{M}{L^3}\right] = \frac{kg}{m^3}$$

$$[g] = \left[\frac{L}{t^2}\right] = \frac{m}{s^2}$$

$$[h] = [L] = m$$ m=3

n−m=1

$$\Pi = \Delta P \cdot \rho^a g^b h^c$$

$$1 = \left(\frac{kg}{m \cdot s^2}\right)\left(\frac{kg}{m^3}\right)^a \left(\frac{m}{s^2}\right)^b (m)^c$$

$$kg: \quad 1 + a = 0$$

$$m: \quad -1 - 3a + b + c = 0$$

$$s: \quad -2 - 2b = 0$$

$$a = b = c = -1$$

$$\Pi = \Delta P \cdot \rho^{-1} g^{-1} h^{-1}$$

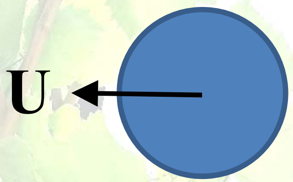
$$F(\Pi) = 0$$

$$\Rightarrow \Pi = \text{constant}$$

$$\frac{\Delta P}{\rho g h} = \text{constant}$$

$$\Delta P \propto \rho g h$$

Example 2:



Wanted: drag acting on a moving sphere in a stationary fluid

$$F = f(\mathrm{D}, \mathrm{U}, \rho, \mu)$$

n=5

$$[F] = N = kg \cdot \frac{m}{s^2}$$

m=3

$$[D] = [L] = m$$

n−m=2

$$[U] = \left[\frac{L}{t}\right] = \frac{m}{s}$$

$$[\rho] = \left[\frac{M}{L^3}\right] = \frac{kg}{m^3}$$

$$[\mu] = \left[\frac{M}{Lt}\right] = \frac{kg}{m \cdot s}$$

$$\Pi_1 = F \cdot \rho^{a_1} D^{b_1} U^{c_1}$$

$$= F / \rho U^2 D^2$$

$$\Pi_2 = \mu \cdot \rho^{a_2} D^{b_2} U^{c_2}$$

$$= \mu / \rho U D$$

$$F / \rho U^2 D^2 = f(\mu / \rho U D)$$

Dimensional Analysis - Buckingham Pi Theorem

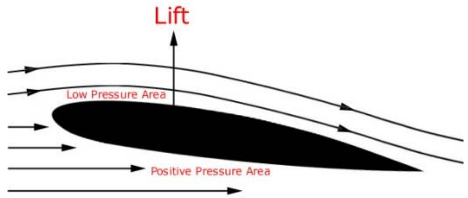
$$\frac{F}{\rho U^2 D^2} = f\left(\frac{\mu}{\rho U D}\right)$$ unknown, determined by experiments

- investigate the effect of different values of $\mu/\rho UD$ on $F/\rho U^2 D^2$ instead of effects of individual parameter ρ, U, D, or μ
- goal 1 (reduce number of investigated parameters)
- goal 2 (model flow vs. real flow)

- Two flows may be involved with different ρ, U, D, or μ but have the same value of $\mu/\rho UD$
  ⇨ must have the same value of $F/\rho U^2 D^2$

$$\left(\frac{\mu}{\rho UD}\right)_{\text{model}} = \left(\frac{\mu}{\rho UD}\right)_{real} \Rightarrow \left(\frac{F}{\rho U^2 D^2}\right)_{\text{model}} = \left(\frac{F}{\rho U^2 D^2}\right)_{real}$$

Example 3:

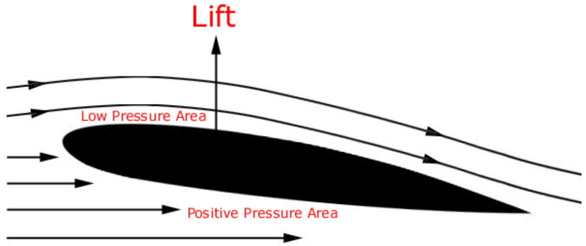


parameter	symbol	unit
Lift per span	L	N/m=kg/s^2
Angle of attack	α	
size of body (e.g. chord)	c	m
Freestream velocity	U_∞	m/s
Freestream density	ρ_∞	kg/m^3
Freestream viscosity	μ_∞	kg/m.s
Freestream speed of sound	a_∞	m/s
gravity	g	m/s^2

n=8
m=3

5Π 's!

Example 3:



$$\Pi_1 = \frac{L}{\frac{1}{2}\rho_\infty U_\infty^2 c} \equiv C_L = \text{lift coefficient}$$

$$\Pi_2 = \alpha = \text{angle of attack}$$

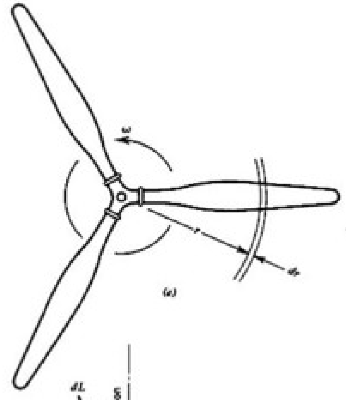
$$\Pi_3 = \frac{\rho_\infty U_\infty c}{\mu_\infty} = \text{Re} = \text{Reynolds number}$$

$$\Pi_4 = \frac{U_\infty}{a_\infty} = Ma = \text{Mach number}$$

$$\Pi_5 = \left(\frac{U_\infty^2}{gc}\right)^{1/2} = \text{Froude \#} = Fr$$

$$C_L = C_L\left(\alpha, \text{Re}, Ma, Fr\right)$$

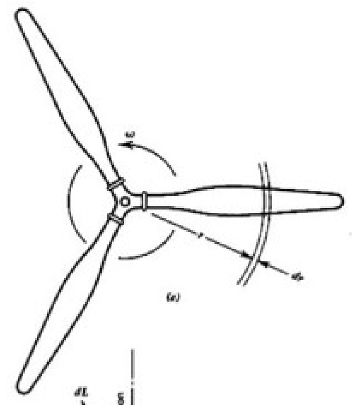
Example 4:



	parameter	symbol	unit
	Thrust	T	N=kg.m/s^2
n=6	Propeller diameter	D	m
m=3	Propeller speed	n	1/s
3Π 's!	Flight speed	V_0	m/s
	Freestream density	ρ_∞	kg/m^3
	Freestream viscosity	μ_∞	kg/m.s

Okulov V.L., Sorensen J.N ., van Kuik G.A.M. Development of the optimum rotor theories. Moscow-Izhevsk: R&C Dyn., 2013. 120 p. ISBN 978-5-93972-957-4.
was translated in English of by interpreters of Institute Termophysics, Novosibirsk, Russia

Example 4:



$$\Pi_1 = \frac{T}{\frac{1}{2}\rho_\infty n^2 D^4} \equiv c_T = \text{thrust coefficient}$$

$$\Pi_2 = \frac{\rho_\infty D^2 n}{\mu_\infty} \sim \frac{\rho_\infty D V_{tip}}{\mu_\infty} = \text{Re} = \text{Reynolds number}$$

$$\Pi_3 = \frac{V_0}{nD} = J = \text{advance ratio}$$

$$c_T = c_T\left(\text{Re}, J\right)$$

Geometric similarity: **(length scale)**

- model and prototype be the same shape and all linear dimensions

f the model be related to corresponding dimensions of the prototype by a constant scale factor.

Kinematic similarity: **(length scale+time scale)**

- velocities at corresponding points are in the same direction and

are related in magnitude by a constant scale factor.

⇨ streamline patterns related by a constant scale factor

Dynamic similarity: **(length scale + time scale+ force scale)**

- two flows have force distributions such that identical types of forces are parallel and are related in magnitude by a constant scale factor at all corresponding points.

To achieve “**Dynamic similarity**” between a real flow and its model flow, all but one of these Π–parameters must be duplicated.

$$F(\Pi_1, \Pi_2, \ldots\ldots, \Pi_{n-m}) = 0$$

Π_1: to be determined

$\Pi_2, \ldots\ldots, \Pi_{n-m}$: same for both the real flow and the model flow

Only if

$$(\Pi_2)_{\text{model}} = (\Pi_2)_{real}$$

$$(\Pi_3)_{\text{model}} = (\Pi_3)_{real}$$

$$\vdots$$

$$(\Pi_{n-m})_{\text{model}} = (\Pi_{n-m})_{real}$$

then

$$(\Pi_1)_{\text{model}} = (\Pi_1)_{real}$$

In the lab, to ensure dynamic similarity, i.e.

$$\vec{F}(x, y, z)_{\text{model}} \propto \vec{F}(x_c, y_c, z_c)_{real}$$

one requires

corresponding point

geometric similarity

and kinematic similarity $\vec{u}(x, y, z)_{\text{model}} \propto \vec{u}(x_c, y_c, z_c)_{\text{real}}$

everywhere

Remark: At least make important Π's in the same; others are made up in some other ways such as analysis, experimental measurement, etc. Reasonable results can be still possible.

1.6 Dimensionless parameters

$$\text{inertial force per unit volume} \sim \rho\, du/dt \sim \rho \frac{U}{L/U} \sim \rho \frac{U^2}{L}$$

$$\text{pressure force per unit volume} \sim \frac{A \Delta P}{A \cdot L} \sim \frac{\Delta P}{L}$$

$$\text{friction force per unit volume} \sim \frac{A \cdot \tau_{xy}}{A \cdot L} \sim \frac{\mu \frac{\partial u}{\partial y}}{L} \sim \frac{\mu U}{L^2}$$

$$\text{gravity force per unit volume} \sim \rho g$$

inertial force $\sim \rho U^2 / L$

pressure force $\sim \Delta P / L$

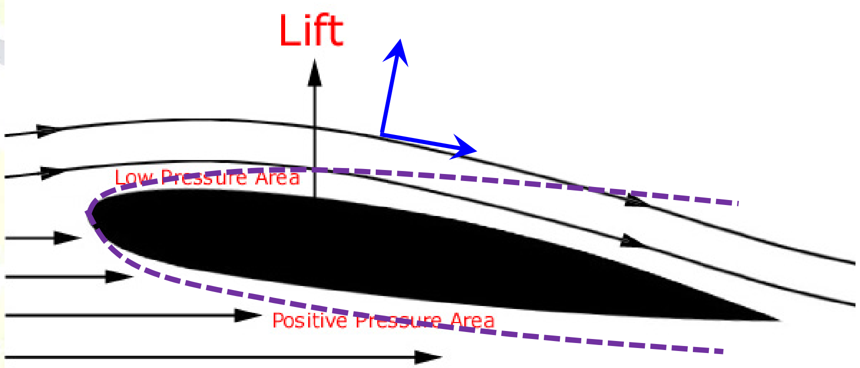
friction force $\sim \mu U / L^2$

gravity force $\sim \rho g$

1.6 Dimensionless parameters

(i) **Reynolds number** $= Re \equiv \dfrac{\text{inertial effect}}{\text{viscous effect}} \equiv \dfrac{\rho U^2/L}{\mu U/L^2} = \dfrac{\rho U L}{\mu} = \dfrac{L^2/\nu}{L/U}$

$Re \ll 1$: viscous diffusion speed >> convection speed



viscous effect >> inertial effect

⇨ ignore convective term

⇨ Stokes flows

$Re \gg 1$: convection speed >> viscous diffusion speed

As $Re \rightarrow 0$, can we ignore viscous force? **No!!**

The larger Re, the thinner region (boundary layer) is affected by the viscous effect.

cases of $\mu \rightarrow 0 \neq$ cases of $\mu=0$

i.e. The case $\mu=0$ is a singularity

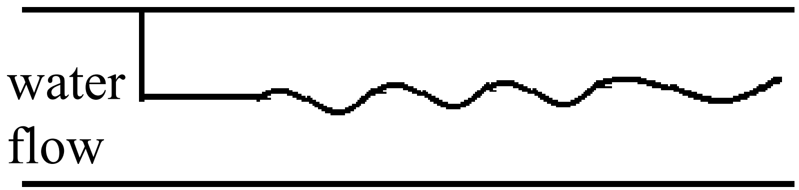
laminar vs turbulent

Reynolds experiments: fixed diameter of the pipe

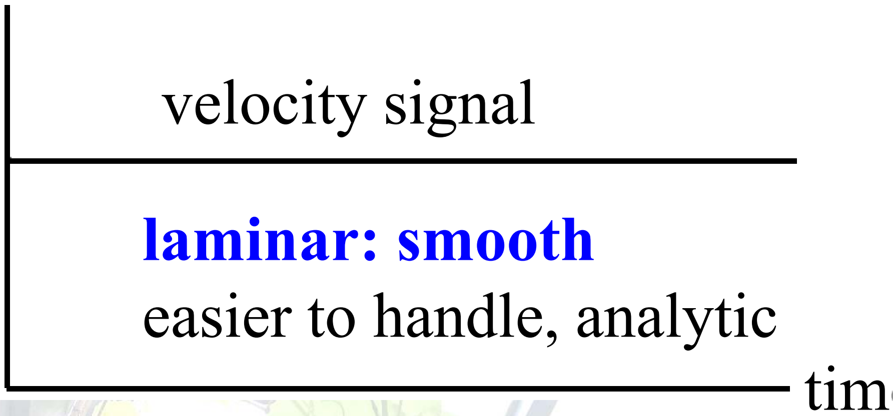
small velocity

dye remains in a single filament
little dispersion little mixing

large velocity

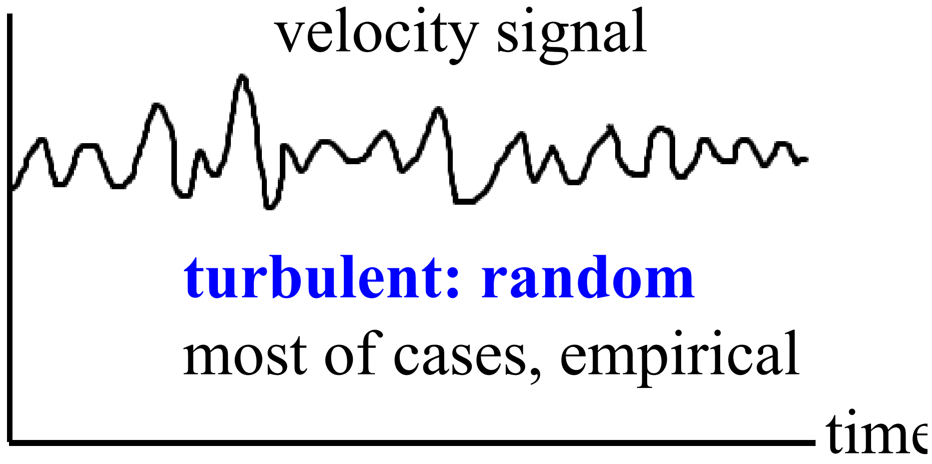


dye stretched, twisted breaks
strong dispersion, strong
mixing



laminar: smooth
easier to handle, analytic

small $Re \equiv \frac{UL}{\nu}$



turbulent: random
most of cases, empirical

large $Re \equiv \frac{UL}{\nu}$

1.6 Dimensionless parameters

(ii) **Mach number** $= M \equiv \dfrac{\text{flow speed}}{\text{sound speed}} = \dfrac{U}{a}$

sound speed $a = \sqrt{\dfrac{dP}{d\rho}}$

$$M^2 = \frac{U^2}{(dP/d\rho)} = \frac{\rho U^2 L^2}{\rho(dP/d\rho)L^2} = \frac{\rho U^2/L \cdot L^3}{\rho(dP/d\rho)L^2}$$

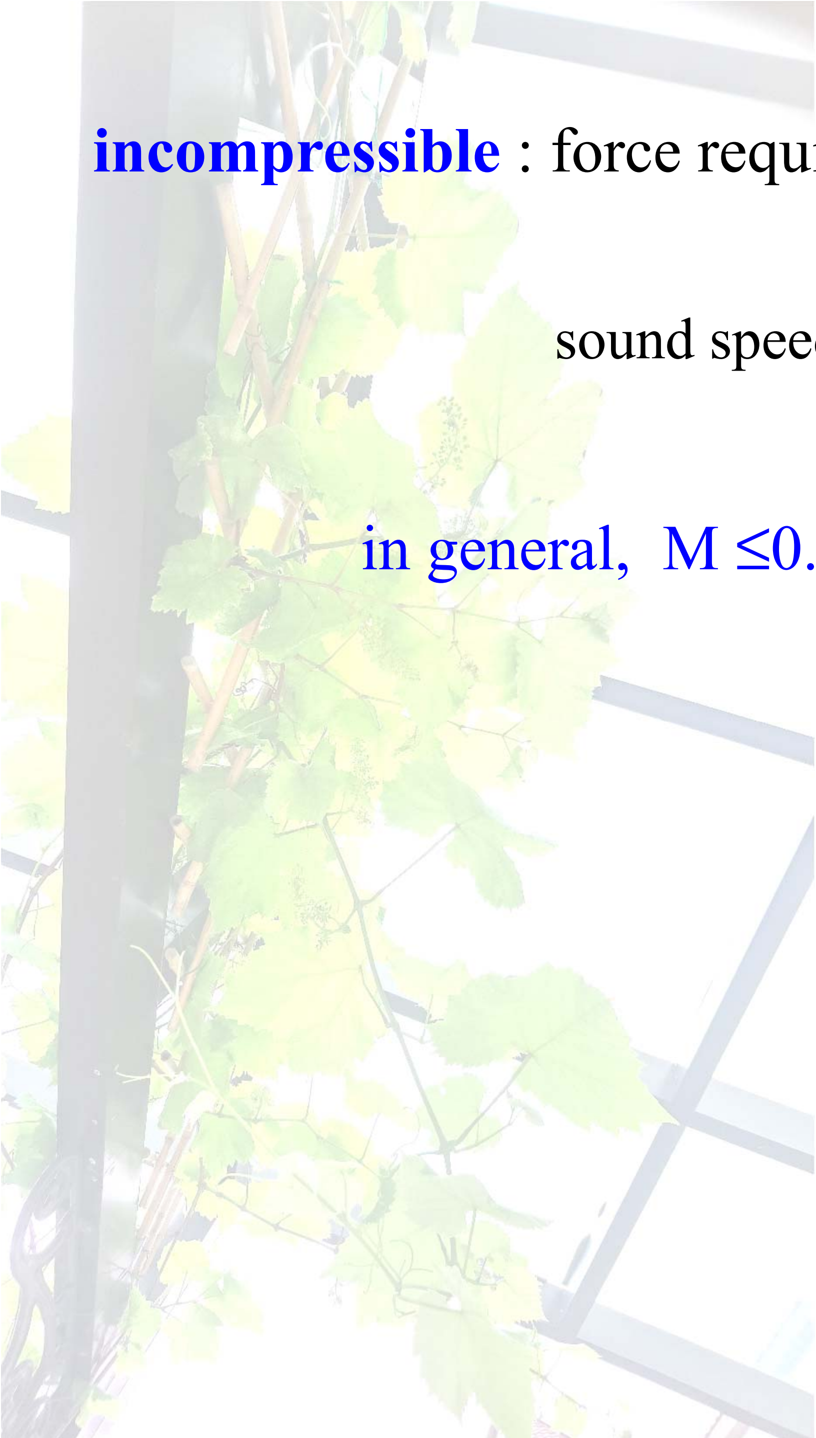
$$= \frac{\text{inertial force}}{\text{force required for compressibility}}$$

inertial force $\sim \rho U^2/L$

pressure force $\sim \Delta P/L$

gravity force $\sim \rho g$

friction force $\sim \mu U/L^2$



incompressible : force required for compressibility >>1

$$\text{sound speed } a = \sqrt{\frac{dP}{d\rho}} >> 1 \quad \Rightarrow \boldsymbol{M} << 1$$

in general, $M \leq 0.3$ ⇨ approximately incompressible

subsonic flow: M<1

sonic flow : M=1

supersonic flow : M>1

hypersonic flow: M>5

1.6 Dimensionless parameters

(iii) **Euler number** $Eu \equiv \dfrac{\text{pressure force}}{\text{inertial force}} = \dfrac{\Delta P / L}{\frac{1}{2}\rho U^2 / L} = \dfrac{\Delta P}{\frac{1}{2}\rho U^2}$

also called"pressure coefficient "(Cp)

(iv) **cavitation number** $= Ca \equiv \dfrac{P - P_v}{\frac{1}{2}\rho U^2}$

P_v = vapor pressure of the liquid fluid

(v) **Froude number** $= Fr = \left(\dfrac{\text{inertial force}}{\text{gravity force}}\right)^{\frac{1}{2}}$

$$= \left(\frac{\rho U^2 / L}{\rho g}\right)^{\frac{1}{2}} = \left(\frac{U^2}{gL}\right)^{\frac{1}{2}} = \frac{U}{\sqrt{gL}}$$

1.6 Dimensionless parameters

(vi) **Weber number** $= We = \dfrac{\text{inertial force}}{\text{surface tension force}} = \dfrac{(\rho U/L)\cdot L^3}{\sigma \cdot L} = \dfrac{\rho U L}{\sigma}$

$\sigma =$ surface tension force per unit length

incompressible viscous flow
- internal flows
  - laminar
  - turbulent
- external flows
  - laminar
  - turbulent

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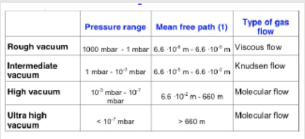
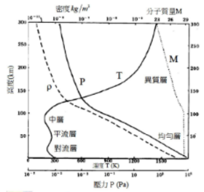
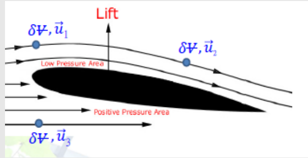
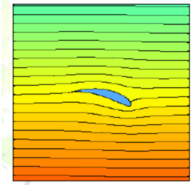
1.			HELDERPAD / Gas Flow Conductance / Vacuum Pressure Ranges， https://helderpad.com/2017/03/02/gas-flow-conductance/ ， 本網站係以著作權法第 52、65 條合理使用本件作品，2022/09/21。
2.			來源：熊年祿等《電離層物理概論》，武漢大學出版社，1999.5， 本網站係以著作權法第 52、65 條合理使用本件作品。
3.			NASA / Student Airfoil Interactive， https://www1.grc.nasa.gov/beginners-guide-to-aeronautics/foilsimstudent/ ， 本網站係以著作權法第 52、65 條合理使用本件作品，2022/09/21。
4.			THE STREAMLINES， https://www.av8n.com/irro/profilo1_e.html ， 本網站係以著作權法第 52、65 條合理使用本件作品，2022/09/21。

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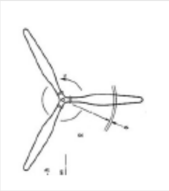
5.			Okulov V.L., Sorensen J.N ., van Kuik G.A.M. Development of the optimum rotor theories. Moscow-Izhevsk: R&C Dyn., 2013. 120 p. ISBN 978-5-93972-957-4. was translated in English of by interpreters of Institute Termophysics, Novosibirsk, Ru5s3sia， 本網站係以著作權法第 52、65 條合理使用本件作品。