Exercises for Unit 4

1. Let (Ω, Σ, μ) be a measure space, it is called a complete measure space if every μ-null set is in Σ. Define

$$\overline{\Sigma} = \{A \subset \Omega : \text{ there are sets } B \text{ and } C \text{ in } \Sigma \text{ such that } C \subset A \subset B \text{ and } \mu(B \setminus C) = 0\};$$

and for $A \in \overline{\Sigma}$ let $\overline{\mu}(A) = \mu(B) = \mu(C)$ where B, C in Σ are as in the definition of $\overline{\Sigma}$. Clearly $\overline{\mu}$ is well-defined on $\overline{\Sigma}$ and is an extension of μ to $\overline{\Sigma}$.

   (i) Show that $\overline{\Sigma}$ is a σ-algebra on Ω and $\overline{\mu}$ is σ-additive on $\overline{\Sigma}$ ($(\Omega, \overline{\Sigma}, \overline{\mu})$ is called the completion of (Ω, Σ, μ)).

   (ii) Show that (Ω, Σ, μ) is complete if and only if

$$(\Omega, \Sigma, \mu) = (\Omega, \overline{\Sigma}, \overline{\mu});$$

   and that $(\Omega, \overline{\Sigma}, \overline{\mu})$ is the smallest complete measure space containing (Ω, Σ, μ) in the sense that if $(\Omega, \hat{\Sigma}, \hat{\mu})$ is a complete measure space with $\Sigma \subset \hat{\Sigma}$ and $\hat{\mu}|_{\Sigma} = \mu$, then $\overline{\Sigma} \subset \hat{\Sigma}$ and $\overline{\mu}(A) = \hat{\mu}(A)$ for $A \in \overline{\Sigma}$.

2. Show that for k, l in $\mathbb{N}$, $(\mathbb{R}^{k+l}, \mathscr{L}^{k+l}, \lambda^{k+l})$ is the completion of $(\mathbb{R}^k \times \mathbb{R}^l, \mathscr{L}^k \otimes \mathscr{L}^l, \lambda^k \times \lambda^l)$.

3. Suppose that $\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are σ-finite complete measure spaces and let μ be the measure constructed from $\mu_1 \times \mu_2$ on $\Sigma_1 \otimes \Sigma_2$ by Method I. Show that $(\Omega_1 \times \Omega_1, \Sigma^{\mu}, \mu)$ is the completion of $(\Omega_1 \times \Omega_2, \Sigma_1 \otimes \Sigma_2, \mu_1 \times \mu_2)$.

4. Let (Ω, Σ, μ) be a σ-finite measure space and f a nonnegative Σ-measurable function on Ω. Put $G_f = \{(w, y) \in \Omega \times [0, \infty) : 0 < y < f(w)\}$. Show that $G_f \in \Sigma \otimes \mathscr{B}$ and $\mu \times \lambda(G_f) = \int_\Omega f d\mu$.

5. Show that $\int_0^\infty (\sum_{j=1}^\infty e^{-jx} \sin x) dx = \sum_{j=1}^\infty \int_0^\infty e^{-jx} \sin x dx$ and use this fact to show that $\int_0^\infty \frac{\sin x}{e^x - 1} dx = \sum_j \frac{1}{1+j^2}$.

6. Show that $\int_0^\infty (\frac{\tan^{-1} t}{t})^2 dt = \pi \ln 2$ by integrating the triple integral

$$\int_0^1 (\int_0^1 (\int_0^\infty \frac{1}{1+x^2t^2} \frac{1}{1+y^2t^2} dt) dx) dy$$